MODELING OF CONCRETE MATERIALS AND STRUCTURES

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Class Meeting #1: Fundamentals

Notation: *Direct and indicial tensor formulations*

Fundamentals: *Stress and Strain (Voigt format)*

Scope: *Linear Elasticity and Triaxial Failure Criteria*

NOTATION

3-D Tensor Format:

1. Scalar Fields:
   Temperature $T(\boldsymbol{x}, t) = T(x_i, t)$ where $i = 1, 2, 3$
2. Vector Fields:
   Displacement $\boldsymbol{u}(\boldsymbol{x}, t) = u_i(x_i, t)$ where $i = 1, 2, 3$
3. 2nd Order Tensor Fields:
   Stress $\boldsymbol{\sigma}(\boldsymbol{x}, t) = \sigma_{ij}(x_i, t)$ where $i, j = 1, 2, 3$
   Strain $\boldsymbol{\epsilon}(\boldsymbol{x}, t) = \epsilon_{ij}(x_i, t)$ where $i, j = 1, 2, 3$

Direct Notation:

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\epsilon}(\boldsymbol{x}, t) = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

DIRECT TENSOR NOTATION

Elastic Stiffness: $\boldsymbol{E}(\boldsymbol{x}, t) = E_{ijkl}(x_i, t)$

Elastic Stress-Strain Relation: $\boldsymbol{\sigma} = \boldsymbol{E} : \boldsymbol{\epsilon}$ such that $\sigma_{ij} = E_{ijkl}\epsilon_{kl}$

Matrix Format of 2-D Stress-Strain Relationship:

$$\left[\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \hline \sigma_{12} \\ \sigma_{21} \end{array}\right] = \left[\begin{array}{cc|cc} E_{1111} & E_{1122} & E_{1112} & E_{1121} \\ E_{2211} & E_{2222} & E_{2212} & E_{2221} \\ \hline E_{1211} & E_{1222} & E_{1212} & E_{1221} \\ E_{2111} & E_{2122} & E_{2112} & E_{2121} \end{array}\right] \left[\begin{array}{c} \epsilon_{11} \\ \epsilon_{22} \\ \hline \epsilon_{12} \\ \epsilon_{21} \end{array}\right]$$

When $\epsilon_{12} = \epsilon_{21}$ and $\sigma_{12} = \sigma_{21}$, the stress-strain relationship may be compacted into $[3 \times 3]$ matrix form since $E_{1112} = E_{1121}$ and $E_{1211} = E_{2111}$:

$$\left[\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \hline \sigma_{12} \end{array}\right] = \left[\begin{array}{cc|c} E_{1111} & E_{1122} & E_{1112} \\ E_{2211} & E_{2222} & E_{2212} \\ \hline E_{1211} & E_{1222} & E_{1212} \end{array}\right] \left[\begin{array}{c} \epsilon_{11} \\ \epsilon_{22} \\ \hline 2\epsilon_{12} \end{array}\right]$$

VOIGT NOTATION [1924]

Linear Elastic Relationship: Triclinic materials exhibit 21 elastic moduli.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \left[\begin{array}{ccc|ccc} E_{1111} & E_{1122} & E_{1133} & E_{1112} & E_{1123} & E_{1131} \\ E_{2211} & E_{2222} & E_{2233} & E_{2212} & E_{2223} & E_{2231} \\ E_{3311} & E_{3322} & E_{3333} & E_{3312} & E_{3323} & E_{3331} \\ \hline E_{1211} & E_{1222} & E_{1233} & E_{1212} & E_{1223} & E_{1231} \\ E_{2311} & E_{2322} & E_{2333} & E_{2312} & E_{2323} & E_{2331} \\ E_{3111} & E_{3122} & E_{3133} & E_{3112} & E_{3123} & E_{3131} \end{array}\right] \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \hline 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{bmatrix}$$

Summary of Voigt Notation:

1. Stress: if $\sigma_{ij} = \sigma_{ji}$ then $[\boldsymbol{\sigma}]_{6\times 1} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^t$
2. Strain: if $\epsilon_{ij} = \epsilon_{ji}$ then $[\boldsymbol{\epsilon}]_{6\times 1} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{31}]^t$
3. Elasticity Matrix: if $\boldsymbol{E} = \boldsymbol{E}^t$ or $E_{ijkl} = E_{klij}$ then $[\boldsymbol{E}]_{6\times 6}$
4. Elastic Strainenergy: $W = \frac{1}{2}\boldsymbol{\epsilon} : \boldsymbol{E} : \boldsymbol{\epsilon} = \frac{1}{2}\epsilon_{ij}E_{ijkl}\epsilon_{kl} = \frac{1}{2}[\boldsymbol{\epsilon}]^t[\boldsymbol{E}][\boldsymbol{\epsilon}]$

CLASSES OF ELASTIC SYMMETRY

1. Monoclinic Materials:

One plane of symmetry with 13 elastic moduli

$$[\boldsymbol{E}] = \left[\begin{array}{ccc|ccc} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & E_{1131} \\ E_{2211} & E_{2222} & E_{2233} & 0 & 0 & E_{2231} \\ E_{3311} & E_{3322} & E_{3333} & 0 & 0 & E_{3331} \\ \hline 0 & 0 & 0 & E_{1212} & E_{1223} & 0 \\ 0 & 0 & 0 & E_{2312} & E_{2323} & 0 \\ E_{3111} & E_{3122} & E_{3133} & 0 & 0 & E_{3131} \end{array}\right]$$

2. Orthotropic Materials:

Two planes of symmetry with 9 elastic moduli

$$[\boldsymbol{E}] = \left[\begin{array}{ccc|ccc} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{2211} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{3311} & E_{3322} & E_{3333} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & E_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{3131} \end{array}\right]$$

CLASSES OF ELASTIC SYMMETRY

3. Transversely Isotropic Materials:

One additional plane of isotropy and 5 elastic moduli

$E_{1111} = E_{2222}$, $E_{1133} = E_{2233}$, $E_{3131} = E_{2323}$, $E_{1212} = 0.5[E_{1111} - E_{1122}]$.

$$[\boldsymbol{E}] = \left[\begin{array}{ccc|ccc} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{2211} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{3311} & E_{3322} & E_{3333} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & E_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{3131} \end{array}\right]$$

4. Cubic Materials

Two additional planes of isotropy and 3 elastic moduli

$E_{1111} = E_{2222} = E_{3333}$, $E_{1122} = E_{1133} = E_{2233}$, $E_{3131} = E_{2323} = E_{1212}$.

$$[\boldsymbol{E}] = \left[\begin{array}{ccc|ccc} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{2211} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{3311} & E_{3322} & E_{3333} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & E_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{3131} \end{array}\right]$$

ISOTROPIC ELASTICITY

Infinite number of planes with equal material properties and 2 elastic moduli

$E_{1111} = E_{2222} = E_{3333}$, $E_{1122} = E_{1133} = E_{2233}$,

$E_{1212} = E_{3131} = E_{2323} = 0.5[E_{1111} - E_{1122}]$.

$$[\boldsymbol{E}] = \left[\begin{array}{ccc|ccc} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{2211} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{3311} & E_{3322} & E_{3333} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & E_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{3131} \end{array}\right]$$

Lamé format of isotropic elastic stress-strain relationship:

$\boldsymbol{\sigma} = \boldsymbol{E} : \boldsymbol{\epsilon}$ where $\boldsymbol{E} = \Lambda \mathbf{1} \otimes \mathbf{1} + 2G\boldsymbol{I}$ and $\boldsymbol{\sigma} = \Lambda(tr\boldsymbol{\epsilon})\mathbf{1} + 2G\boldsymbol{\epsilon}$

where $\Lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}$ and $G = \frac{E}{2(1+\nu)}$

$$[\boldsymbol{E}] = \left[\begin{array}{ccc|ccc} \Lambda + 2G & \Lambda & \Lambda & & & \\ \Lambda & \Lambda + 2G & \Lambda & & 0 & \\ \Lambda & \Lambda & \Lambda + 2G & & & \\ \hline & & & G & & \\ & 0 & & & G & \\ & & & & & G \end{array}\right]$$

ISOTROPIC ELASTICITY

Elastic Compliance Relationship:

$$\boldsymbol{\epsilon} = \boldsymbol{C} : \boldsymbol{\sigma} \quad \text{where} \quad \boldsymbol{C} = -\frac{\nu}{E}\mathbf{1} \otimes \mathbf{1} + \frac{1}{2G}\boldsymbol{I} \quad \text{and} \quad \boldsymbol{\epsilon} = -\frac{\nu}{E}(tr\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2G}\boldsymbol{\sigma}$$

Matrix Format of 3-D Strain-Stress Relationship:

$$[\boldsymbol{C}] = [\boldsymbol{E}^{-1}] = \left[\begin{array}{ccc|ccc} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{3131} \end{array}\right]$$

Poisson's ratio: $\nu = -\frac{\epsilon_{lat}}{\epsilon_{axial}}$

$$[\boldsymbol{C}] = [\boldsymbol{E}^{-1}] = \frac{1}{E}\left[\begin{array}{ccc|ccc} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & 0 & \\ -\nu & -\nu & 1 & & & \\ \hline & & & 2[1+\nu] & & \\ & 0 & & & 2[1+\nu] & \\ & & & & & 2[1+\nu] \end{array}\right]$$

ISOTROPIC ELASTICITY

Volumetric-Deviatoric Split: $\boldsymbol{\epsilon} = \boldsymbol{e} + \frac{1}{3}(tr\boldsymbol{\epsilon})\mathbf{1}$ and $\boldsymbol{\sigma} = \boldsymbol{s} + \frac{1}{3}(tr\boldsymbol{\sigma})\mathbf{1}$

(a) Volumetric Response:

$$\boxed{(tr\boldsymbol{\sigma}) = 3\,K(tr\boldsymbol{\epsilon})}$$

where $K = \Lambda + \frac{2}{3}G = \frac{E}{3(1-2\nu)}$

(b) Deviatoric Response:

$$\boxed{\boldsymbol{s} = 2\,G\,\boldsymbol{e}}$$

where $G = \frac{3(1-2\nu)}{2(1+\nu)}K = \frac{E}{2(1+\nu)}$

Canonical decoupling of strainenergy: $W = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\epsilon} = \frac{1}{2}K(tr\boldsymbol{\epsilon})^2 + G\boldsymbol{e} : \boldsymbol{e}$

Constitutive Restrictions: $0 < E < \infty$ and $-1 < \nu < 0.5$.

FAILURE HYPOTHESES

1. Isotropic Strength Criteria: $F_{iso}(\boldsymbol{\sigma}) = F_{iso}(\boldsymbol{R}^t \cdot \boldsymbol{\sigma} \cdot \boldsymbol{R})$

(a) Principal Format: $F_{iso} = F_{iso}(\sigma_1, \sigma_2, \sigma_3) = 0$

(b) Invariant Format: $F_{iso} = F_{iso}(I_1, I_2, I_3)_\sigma = 0$
where $I_1 = (tr\boldsymbol{\sigma})$; $I_2 = \frac{1}{2}(tr\boldsymbol{\sigma}^2)$; $I_3 = \frac{1}{3}(tr\boldsymbol{\sigma}^3)$

(c) Volumetric-Deviatoric Format: $F_{iso} = F_{iso}(I_1, J_2, J_3)_\sigma = 0$
where $J_1 = 0$; $J_2 = \frac{1}{2}(tr\boldsymbol{s}^2)$; $J_3 = \frac{1}{3}(tr\boldsymbol{s}^3) = \det \boldsymbol{s}$

Examples: Rankine, Tresca, Mohr-Coulomb, von Mises, Drucker-Prager, Willam-Warnke, Ottosen, Gudehus, Lade ..

2. Anisotropic Strength Criteria: $F_{aniso}(\boldsymbol{\sigma} : \boldsymbol{P} : \boldsymbol{\sigma}, \boldsymbol{q} : \boldsymbol{\sigma}) = 0$

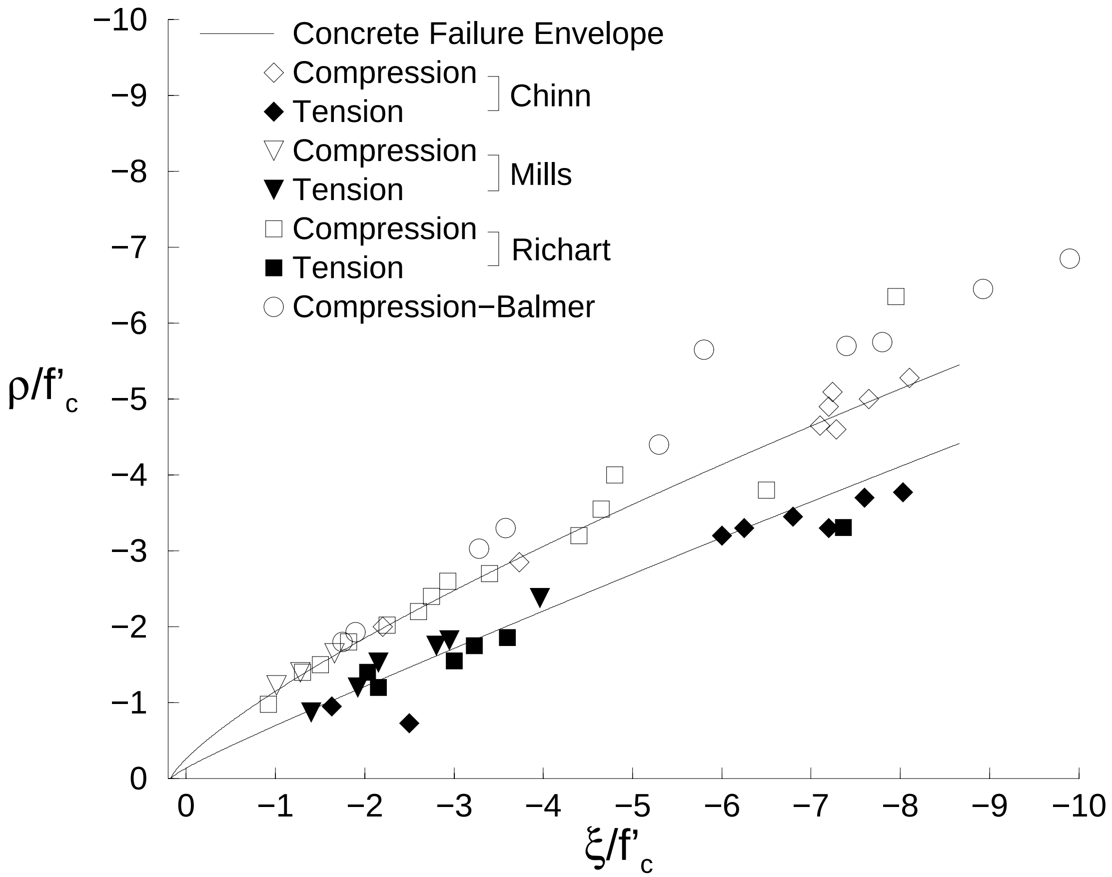
$F_{aniso} = F(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}) = 0$

Examples: Hill, Hoffman, Tsai-Wu ..

GEOMETRIC FAILURE ENVELOPE

Haigh-Westergaard Format of 3-Invariant Format: $F_{iso} = F_{iso}(\xi, \rho, \theta)_\sigma = 0$

- Hydrostatic Coordinate: $\xi = \frac{1}{\sqrt{3}}(tr\boldsymbol{\sigma})$
- Deviatoric Coordinate $\rho = \sqrt{\boldsymbol{s} : \boldsymbol{s}}$
- Angle of Similarity: $cos3\theta = \frac{\sqrt{27}J_3}{2J_2^{1.5}}$



TRIAXIAL FAILURE ENVELOPE FOR CONCRETE

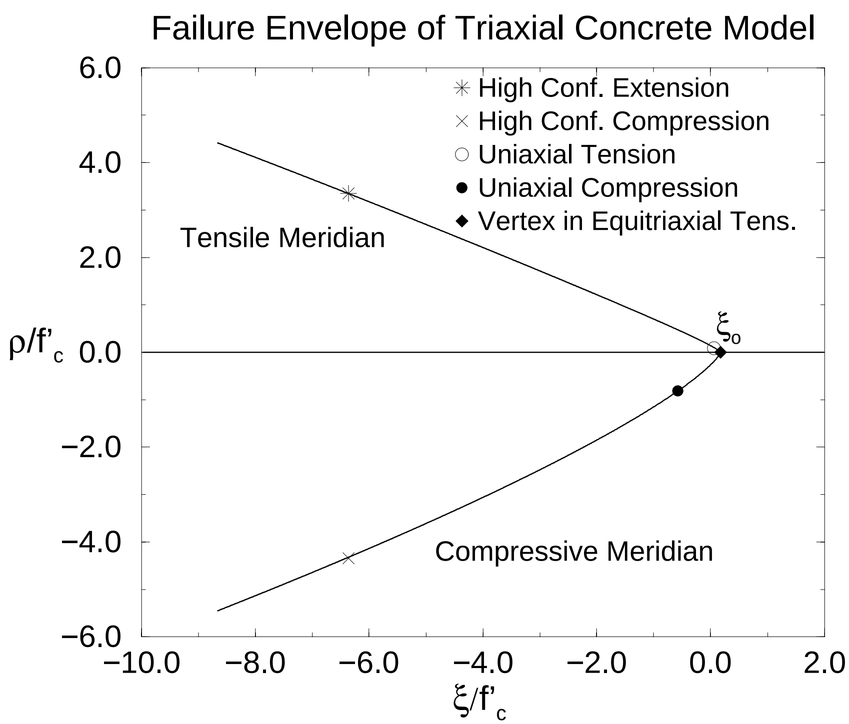
5-Parameter Willam-Warnke Model [1975], 3-Pars. Menétrey-Willam [1995]

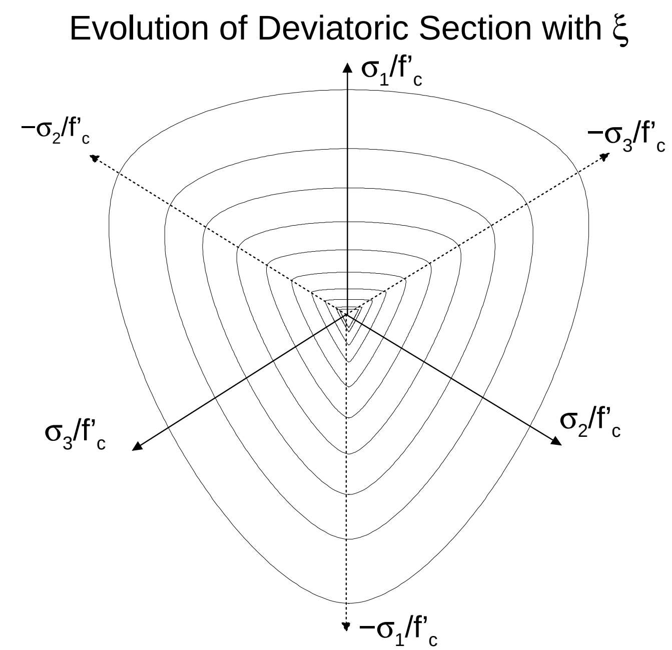
Triaxial Hardening-Softening Model: Kang-Willam [1999]

$$F(\xi,\rho,\theta)_{\text{fail}} = \frac{\rho r(\theta,e)}{f_c'} - \frac{\rho_1}{f_c'}\left[\frac{\xi-\xi_o}{\xi_1-\xi_o}\right]^\alpha = 0$$

Out-of Roundness:

$$r(\theta,e) = \frac{4(1-e^2)cos^2\theta+(2e-1)^2}{2(1-e^2)cos\theta+(2e-1)\sqrt{4(1-e^2)cos^2\theta+5e^2-4e}}$$





ALTERNATIVE FAILURE HYPOTHESES

2. Strain-Based Deformation Criteria: $F(\boldsymbol{\epsilon}) = 0$

Isotropic Case: $F_{iso}(\boldsymbol{\epsilon}) = F_{iso}(\epsilon_1, \epsilon_2, \epsilon_3) = 0$ or $F_{iso}(I_1, I_2, I_3)_\epsilon = 0$ or $F_{iso}(I_1, J_2, J_3)_\epsilon = 0$

Examples: St. Venant ...

3. Energetic Failure Criteria: $F(W) = 0$

Isotropic Case: $F_W(\boldsymbol{\sigma} : \boldsymbol{\epsilon}) = 0$ or $F_{W_{vol}}(tr\boldsymbol{\sigma}\, tr\boldsymbol{\epsilon}) = 0$ or $F_{W_{dev}}(\boldsymbol{s} : \boldsymbol{e}) = 0$

Examples: Beltrami, Huber ...

CONCLUDING REMARKS

Main Lessons from Class #1:

Voigt Notation for Minor Symmetries:
Matrix Form of Elastic Stiffness and Compliance

Canonical Form of Linear Elastic Behavior:
Decoupling of Volumetric-Deviatoric K-G Moduli

Concrete Failure:
Strong Interaction of all Three Invariants