MODELING OF CONCRETE MATERIALS AND STRUCTURES

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Class Meeting #2: Nonlinear Elastic Models

Isotropic Elastic Models: *Invariant vs Principal Formulations*

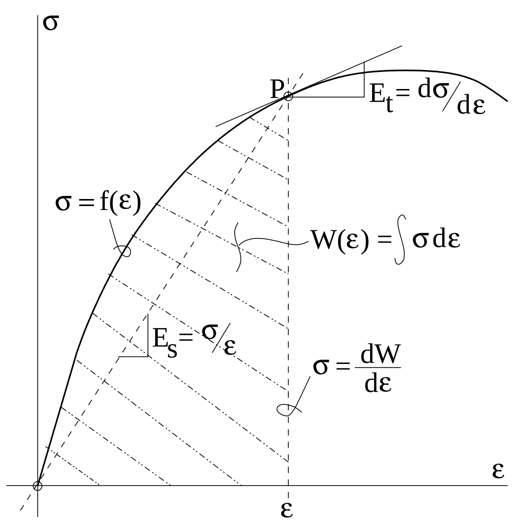
Elastic Damage Models: *Scalar Damage, Volumetric-Deviatoric Damage*

Fixed vs Rotating Crack Models: *Anisotropic vs Isotropic Damage*

NONLINEAR ELASTICITY

Nonlinear Descriptions:

1. Algebraic Format: Pseudo-Elasticity
   $\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{\epsilon})$ reduces to Secant Stiffness Model: $\boldsymbol{\sigma} = \boldsymbol{E}_{sec} : \boldsymbol{\epsilon}$
2. Integral Format: Hyper-Elasticity
   $\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}$ Given $W = W(\boldsymbol{\epsilon})$ Strainenergy Potential $\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}$
3. Differential Format: Hypo-Elasticity
   $\dot{\boldsymbol{\sigma}} = \boldsymbol{g}(\boldsymbol{\sigma}\dot{\boldsymbol{\epsilon}})$ reduces to Tangent Stiffness Model: $\dot{\boldsymbol{\sigma}} = \boldsymbol{E}_{tan} : \dot{\boldsymbol{\epsilon}}$



ALGEBRAIC FORMAT: CAUCHY ELASTICITY

Elastic Stiffness: $\boxed{\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{\epsilon})}$

Representation Theorem of Isotropic Tensor Functions:

$$\boxed{\boldsymbol{\sigma} = \Phi_1 \mathbf{1} + \Phi_2 \boldsymbol{\epsilon} + \Phi_3 \boldsymbol{\epsilon}^2}$$

Three Invariant Response Functions: $\Phi_i = \Phi_i(I_1, I_2, I_3)_\epsilon$

Elastic Secant Relationship: $\boxed{\boldsymbol{\sigma} = \boldsymbol{E}_{sec} : \boldsymbol{\epsilon}}$

Matrix Format of Secant Stiffness: Letting $\Phi_3 = 0$
$\Phi_1 = \Phi_1(I_1)_\epsilon \Rightarrow K_{sec} = K(tr\boldsymbol{\epsilon})$ and $\Phi_2 = \Phi_2(J_2)_\epsilon \Rightarrow G_{sec} = G(tr\boldsymbol{e}^2)$

$$[\boldsymbol{E}]_{sec} = \left[\begin{array}{ccc|ccc} K_s + \frac{4}{3}G_s & K_s - \frac{2}{3}G_s & K_s - \frac{2}{3}G_s & & & \\ K_s - \frac{2}{3}G_s & K_s + \frac{4}{3}G_s & K_s - \frac{2}{3}G_s & & 0 & \\ K_s - \frac{2}{3}G_s & K_s - \frac{2}{3}G_s & K_s + \frac{4}{3}G_s & & & \\ \hline & & & G_s & & \\ & 0 & & & G_s & \\ & & & & & G_s \end{array}\right]$$

SCALAR FORMS OF ELASTIC DAMAGE

Elastic Damage Relationship: $\boxed{\boldsymbol{\sigma} = \boldsymbol{E}^d_{sec} : \boldsymbol{\epsilon}}$

Matrix Format of Secant Stiffness: Letting $\Phi_3 = 0$

(a) Volumetric Damage: $K_d = [1 - d_{vol}]K_0$ with $d_{vol} = 1 - \frac{K_s}{K_0}$

(b) Deviatoric Damage: $G_d = [1 - d_{dev}]G_0$ with $d_{dev} = 1 - \frac{G_s}{G_0}$

$$[\boldsymbol{E}]^d_{sec} = \left[\begin{array}{ccc|ccc} K_d + \frac{4}{3}G_d & K_d - \frac{2}{3}G_d & K_d - \frac{2}{3}G_d & & & \\ K_d - \frac{2}{3}G_d & K_d + \frac{4}{3}G_d & K_d - \frac{2}{3}G_d & & 0 & \\ K_d - \frac{2}{3}G_d & K_d - \frac{2}{3}G_d & K_d + \frac{4}{3}G_d & & & \\ \hline & & & G_d & & \\ & 0 & & & G_d & \\ & & & & & G_d \end{array}\right]$$

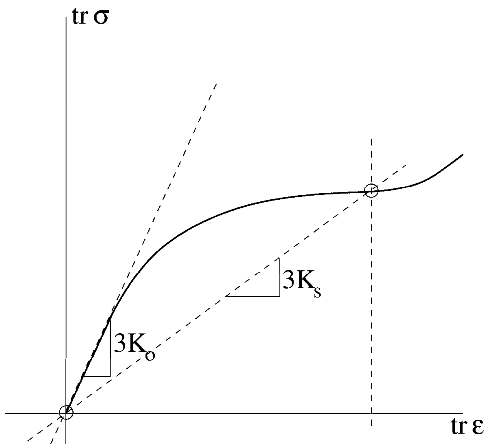
Note: Damage is evolutionary process w/ $\dot{d}_{vol} > 0$ and $\dot{d}_{dev} > 0$.

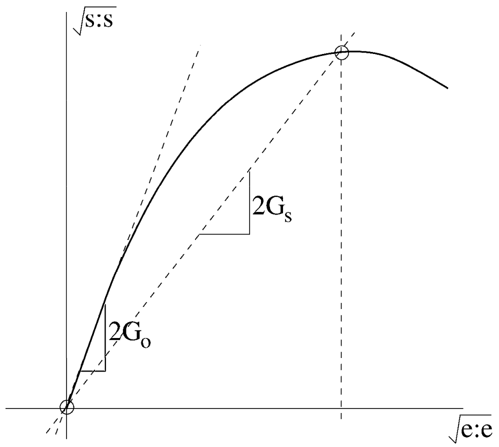
SINGLE SCALAR FORM OF ELASTIC DAMAGE

Scalar Damage Relationship: $\boxed{\boldsymbol{\sigma} = [1-d]\boldsymbol{E}_0 : \boldsymbol{\epsilon}}$

Single scalar damage function: $d = d_{vol} = d_{dev} = 1 - \frac{E_s}{E_0}$ whereby $\nu = \nu_0 = const.$

$$[\boldsymbol{E}]^d_{sec} = [1-d] \left[\begin{array}{ccc|ccc} K_0 + \frac{4}{3}G_o & K_0 - \frac{2}{3}G_0 & K_0 - \frac{2}{3}G_0 & & & \\ K_0 - \frac{2}{3}G_0 & K_0 + \frac{4}{3}G_0 & K_0 - \frac{2}{3}G_0 & & 0 & \\ K_0 - \frac{2}{3}G_0 & K_0 - \frac{2}{3}G_0 & K_0 + \frac{4}{3}G_0 & & & \\ \hline & & & G_0 & & \\ & 0 & & & G_0 & \\ & & & & & G_0 \end{array} \right]$$





INTEGRAL FORM: GREEN ELASTICITY

Hyperelastic Stress: $\boxed{\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}}$

$W(\boldsymbol{\epsilon}) = \int_\epsilon \boldsymbol{\sigma} : d\boldsymbol{\epsilon} = \int_\epsilon \frac{\partial W}{\partial \boldsymbol{\epsilon}} : d\boldsymbol{\epsilon} = \int_\epsilon dW$ with $W(\boldsymbol{\epsilon}) = \oint_\epsilon dW = 0$

Representation Theorem of Isotropic Scalar Functions:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{\epsilon}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{\epsilon}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \boldsymbol{\epsilon}}$$

Three Invariant Response Functions: $\boxed{\boldsymbol{\sigma} = W_1 \mathbf{1} + W_2\, \boldsymbol{\epsilon} + W_3\, \boldsymbol{\epsilon}^2}$

where $W_i = W_i(I_1, I_2, I_3)_\epsilon$ with $\frac{\partial W_i}{\partial I_j} = \frac{\partial W_j}{\partial I_i}$ since $\frac{\partial^2 W}{\partial I_i \partial I_j} = \frac{\partial^2 W}{\partial I_j \partial I_i}$.

Assuming $W_3 = 0$:

$$\boxed{W(\boldsymbol{\epsilon}) = W_{vol}(tr\boldsymbol{\epsilon}) + W_{dev}(tr\boldsymbol{e}^2)}$$

Octahedral Format of Hyperelastic Secant Stiffness:

$\sigma_{oct} = \frac{1}{3} tr\boldsymbol{\sigma}$; $\tau_{oct} = (\frac{1}{3} tr\boldsymbol{s}^2)^{\frac{1}{2}}$

$$\begin{bmatrix} \sigma_{oct} \\ \tau_{oct} \end{bmatrix} = \begin{bmatrix} 3K_s(tr\boldsymbol{\epsilon}) & 0 \\ 0 & 2G_s(tr\boldsymbol{e}^2) \end{bmatrix} \begin{bmatrix} \epsilon_{oct} \\ \gamma_{oct} \end{bmatrix}$$

PRINCIPAL COORDINATE FORMAT OF GREEN ELASTICITY

Hyperelastic Stress: $\sigma_i = \frac{\partial W}{\partial \epsilon_i}$ where $W(\boldsymbol{\epsilon}) = W(\epsilon_1, \epsilon_2, \epsilon_3)$

$$\sigma_1 = \frac{\partial W}{\partial \epsilon_1};\ \ \sigma_2 = \frac{\partial W}{\partial \epsilon_2};\ \ \sigma_3 = \frac{\partial W}{\partial \epsilon_3}$$

Tangential Format: $\boxed{\dot{\sigma}_i = \frac{\partial \sigma_i}{\partial \epsilon_j} \dot{\epsilon}_j = [\frac{\partial^2 W}{\partial \epsilon_i \partial \epsilon_j}] \dot{\epsilon}_j}$

$$\begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \dot{\sigma}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 W}{\partial \epsilon_1 \partial \epsilon_1} & \frac{\partial^2 W}{\partial \epsilon_1 \partial \epsilon_2} & \frac{\partial^2 W}{\partial \epsilon_1 \partial \epsilon_3} \\ \frac{\partial^2 W}{\partial \epsilon_2 \partial \epsilon_1} & \frac{\partial^2 W}{\partial \epsilon_2 \partial \epsilon_2} & \frac{\partial^2 W}{\partial \epsilon_2 \partial \epsilon_3} \\ \frac{\partial^2 W}{\partial \epsilon_3 \partial \epsilon_1} & \frac{\partial^2 W}{\partial \epsilon_3 \partial \epsilon_2} & \frac{\partial^2 W}{\partial \epsilon_3 \partial \epsilon_3} \end{bmatrix} \begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \end{bmatrix}$$

Note $\#1$: Symmetry and 'Apparent Orthotropy' due strain-induced anisotropy.

Note $\#2$: Shear Stiffness maintains co-axiality of principal axes: $\theta^\sigma = \theta^\epsilon$.

$$\begin{bmatrix} \dot{\tau}_{12} \\ \dot{\tau}_{23} \\ \dot{\tau}_{31} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\frac{\sigma_2 - \sigma_1}{\epsilon_2 - \epsilon_1} & 0 & 0 \\ 0 & \frac{1}{2}\frac{\sigma_3 - \sigma_2}{\epsilon_3 - \epsilon_2} & 0 \\ 0 & 0 & \frac{1}{2}\frac{\sigma_1 - \sigma_3}{\epsilon_1 - \epsilon_3} \end{bmatrix} \begin{bmatrix} \dot{\gamma}_{12} \\ \dot{\gamma}_{23} \\ \dot{\gamma}_{31} \end{bmatrix}$$

DIFFERENTIAL FORM: TRUESDELL ELASTICITY

Hypoelastic Stress Format: $\boxed{\dot{\boldsymbol{\sigma}} = \boldsymbol{g}(\boldsymbol{\sigma}, \dot{\boldsymbol{\epsilon}})}$

Incrementally Linear Hypoelastic Formulation:

Tangential Stiffness Format: $\boxed{\dot{\boldsymbol{\sigma}} = \boldsymbol{E}_{tan} : \dot{\boldsymbol{\epsilon}}}$ where $\boldsymbol{E}_{tan} = \boldsymbol{E}(\boldsymbol{\sigma})$

From Representation Theorem of Isotropic Tensor Functions we find:

$$\boldsymbol{E}_{tan} = \begin{bmatrix} C_1 \mathbf{1} \otimes \mathbf{1} & +C_2 \boldsymbol{\sigma} \otimes \mathbf{1} & +C_3 \boldsymbol{\sigma}^2 \otimes \mathbf{1} \\ +C_4 \mathbf{1} \otimes \boldsymbol{\sigma} & +C_5 \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} & +C_6 \boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma} \\ +C_7 \mathbf{1} \otimes \boldsymbol{\sigma}^2 & +C_8 \boldsymbol{\sigma} \otimes \boldsymbol{\sigma}^2 & +C_9 \boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma}^2 \\ +C_{10}[\mathbf{1} \bar{\otimes} \mathbf{1} + \mathbf{1} \underline{\otimes} \mathbf{1}] & +C_{11}[\boldsymbol{\sigma} \bar{\otimes} \mathbf{1} + \mathbf{1} \underline{\otimes} \boldsymbol{\sigma}] & +C_{12}[\boldsymbol{\sigma}^2 \bar{\otimes} \mathbf{1} + \mathbf{1} \underline{\otimes} \boldsymbol{\sigma}^2 \end{bmatrix}$$

Note: Path-Independence requires that hypoelastic constitutive relations satisfy integrability conditions.

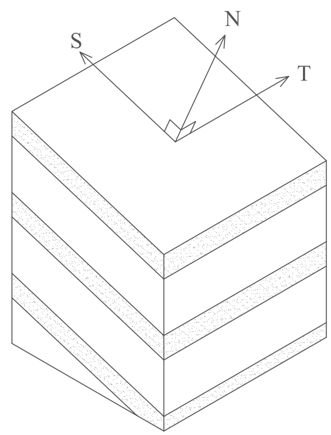
$$\boldsymbol{\sigma} = \int_{\epsilon} \boldsymbol{E}_{tan}(\boldsymbol{\sigma}) : \frac{d\boldsymbol{\epsilon}}{dt} dt$$

SMEARED CRACK APPROACH

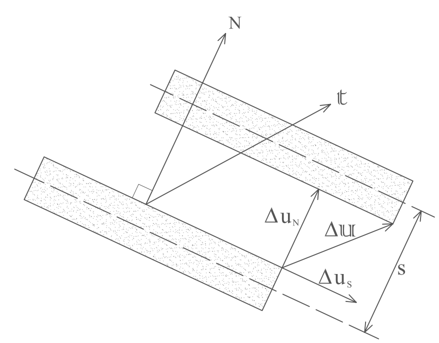
1. Fixed Crack Approach:
Orthotropic material formulation (permanent crack memory)

2. Rotating Crack Approach:
Isotropic material formulation (fading crack memory)
Additional Crack Compliance: due to crack separation $\boldsymbol{\epsilon}^f = \frac{\Delta \boldsymbol{u}}{s}$

$$\boxed{\boldsymbol{\epsilon} = -\frac{\nu}{E}(tr\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2G}\boldsymbol{\sigma} + \frac{1}{2}[\boldsymbol{N} \otimes \boldsymbol{\epsilon}^f + \boldsymbol{\epsilon}^f \otimes \boldsymbol{N}]}$$



Smeared Cracking



Traction - Separation $\mathbb{t}$ - $\Delta\mathbb{u}$

In-plane strains vanish due in local $[\boldsymbol{N}, \boldsymbol{S}, \boldsymbol{T}]$ system such that $\epsilon^f_{SS} = \epsilon^f_{TT} = \epsilon^f_{ST} = 0$, where $\boldsymbol{N}$ is the normal vector to initial crack direction.

1. FIXED CRACK APPROACH

Softening Traction-Separation Model in fixed axes of Orthotropy:

$$t_N = E_N(\epsilon_N, \epsilon_T)\epsilon_N^f \quad \text{and} \quad t_T = E_T(\epsilon_N, \epsilon_T)\epsilon_T^f$$

(a) Interfacial Relations of normal components:

$$\begin{bmatrix} \epsilon_{NN} \\ \epsilon_{SS} \\ \epsilon_{TT} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 + EC_N & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_{NN} \\ \sigma_{SS} \\ \sigma_{TT} \end{bmatrix}$$

(b) Interfacial relations of shear components:

$$\begin{bmatrix} \gamma_{NS} \\ \gamma_{ST} \\ \gamma_{TN} \end{bmatrix} = \frac{1}{G} \begin{bmatrix} 1 + GC_T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + GC_T \end{bmatrix} \begin{bmatrix} \sigma_{NS} \\ \sigma_{ST} \\ \sigma_{TN} \end{bmatrix}$$

Note: Shear retention factor relates $C_T = \frac{1}{E_T(\epsilon_N, \epsilon_T)}$ to $C_N = \frac{1}{E_N(\epsilon_N, \epsilon_T)}$

Compliance Format of Cracking when crack orientation rotates with the principal axes of strain:

$$[\boldsymbol{N}, \boldsymbol{S}, \boldsymbol{T}] \Rightarrow [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$$

(a) Interfacial relations of principal compliances: $C_N = \frac{1}{E_N(\epsilon_N, \epsilon_T)}$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 + EC_N & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_1 \\ \sigma_3 \end{bmatrix}$$

(b) Interfacial relations of shear components:

Argument of 'isotropy' when the principal axes of stress coincide with the principal axes of strain. This requires that the tangential shear compliance follows:

$$\begin{bmatrix} \dot{\gamma}_{12} \\ \dot{\gamma}_{23} \\ \dot{\gamma}_{31} \end{bmatrix} = \begin{bmatrix} 2\frac{\epsilon_2 - \epsilon_1}{\sigma_2 - \sigma_1} & 0 & 0 \\ 0 & 2\frac{\epsilon_3 - \epsilon_2}{\sigma_3 - \sigma_2} & 0 \\ 0 & 0 & 2\frac{\epsilon_1 - \epsilon_3}{\sigma_1 - \sigma_3} \end{bmatrix} \begin{bmatrix} \dot{\tau}_{12} \\ \dot{\tau}_{23} \\ \dot{\tau}_{31} \end{bmatrix}$$

CONCLUDING REMARKS

Main Lessons from Class # 2:

Nonlinear Hyperelasticity:

preserves path-independence, reversibility and energy (no dissipation)

Canonical Form of Nonlinear Elastic Behavior:

Volumetric-Deviatoric Damage Model for $K_s - G_s$

Smeared Cracking of Concrete:

Fixed crack approach (orthotropic format) introduces shear locking -
Rotating crack approach minimizes shear locking (isotropic format)

Rotating Cracking in Form of Tensile Damage due to C_N:

$\boldsymbol{\epsilon} = -\frac{\nu}{E}(tr\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2G}\boldsymbol{\sigma} + C_N\sigma_1[\boldsymbol{e}_1 \otimes \boldsymbol{e}_1]$

Note analogy to plastic softening according to Rankine.