MODELING OF CONCRETE MATERIALS AND STRUCTURES

Kaspar Willam
University of Colorado at Boulder

Class Meeting #3: Elastoplastic Concrete Models

Uniaxial Model: *Strain-Driven Format of Elastoplasticity*

Triaxial Model: *Generalized Format of Elastoplasticity*

Isotropic Hardening/Softening: *Volumetric-Deviatoric Interaction*

Rotating Plastic Crack Model: *Softening Rankine Formulation*

ELASTOPLASTIC MATERIAL MODEL

Fundamental Steps:

1. Additive Decomposition: Elastic-Plastic Partition $\quad \boldsymbol{\epsilon} = \boldsymbol{\epsilon}_e + \boldsymbol{\epsilon}_p$
   Incremental format of elastic stress
   $\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{E}} : [\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_p]$ yields elastoplastic tangent stiffness: $\quad \dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{E}}_{ep} : \dot{\boldsymbol{\epsilon}}$

2. Yield Condition: Plastic Initiation and Persistence: $\quad F(\boldsymbol{\sigma}) = 0$
   Plastic consistency condition distinguishes
   plastic loading from elastic unloading $\dot{F} = \frac{\partial F}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = 0 \quad \Rightarrow \quad \boldsymbol{n} : \dot{\boldsymbol{\sigma}} = \mathbf{0}$

3. Flow Rule: Plastic Evolution Equation $\quad \dot{\boldsymbol{\epsilon}}_p = \dot{\lambda} \boldsymbol{m}$
   Orientation of plastic flow is defined by $\boldsymbol{m} = \frac{\partial Q}{\partial \boldsymbol{\sigma}}$
   and magnitude by plastic multiplier $\dot{\lambda} > 0$

4. Hardening/Softening Rule: Plastic Stiffness $\quad H_p = -\frac{\partial F}{\partial \dot{\lambda}}$
   normally expressed in terms of an invariant
   stress-plastic strain (plastic work) relationship $\quad E_p = \frac{d\sigma_{eq}}{d\epsilon^p_{eq}}$

UNIAXIAL ELASTOPLASTIC MODEL

1. Deformation Theory of Hencky [1924]: Total secant relationship
2. Flow Theory of Prandtl-Reuss [1928]: Incremental tangent relationship

Additive Decomposition:

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_p \quad \text{where} \quad \dot{\epsilon}_e = \frac{\dot{\sigma}}{E} \quad \text{and} \quad \dot{\epsilon}_p = \frac{\dot{\sigma}}{E_p}$$

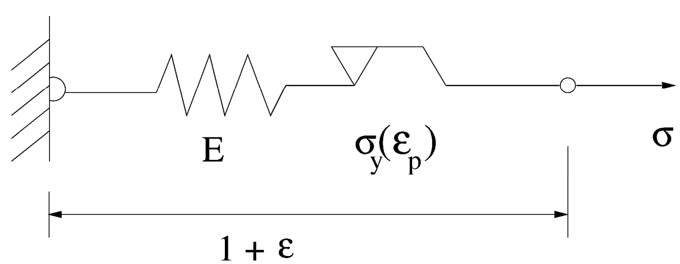
Consequently,

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{\dot{\sigma}}{E_p} = \frac{\dot{\sigma}}{E_{ep}}$$

Elastoplastic Tangent Stiffness Relationship:

$$\dot{\sigma} = E_{ep}\dot{\epsilon} \quad \text{where} \quad E_{ep} = \frac{E E_p}{E + E_P}$$

Note $E_{ep} = \infty$ when $E_p^{crit} = -E$.



UNIAXIAL ELASTOPLASTIC MODEL

Note: $\dot{\epsilon}_p = \frac{\dot{\sigma}}{E_p} = \frac{0}{0}$ when $E_p = 0$

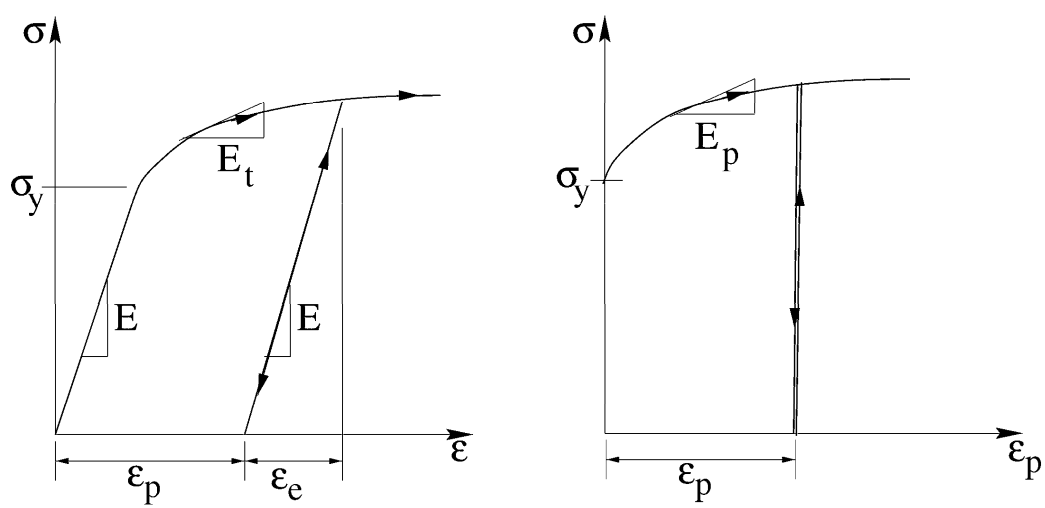
Use “strain” rather than “stress” control:

$$\dot{\epsilon}_p = \frac{\dot{\sigma}}{E_p} = \frac{E}{E + E_p}\dot{\epsilon}$$

Formal Yield Condition: $F(\sigma) = |\sigma| - \sigma_y = 0$

Plastic action

(i) when stress path reaches the yield capacity of the material $|\sigma| = \sigma_y$
(ii) persistent plastic loading when $\frac{dF}{d\sigma}E\dot{\epsilon} > 0$ for strain control.



IDEAL J_2-ELASTOPLASTICITY I

Mises Yield Function:

$$F(\boldsymbol{s}) = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s} - \frac{1}{3}\sigma_y^2 = 0$$

Associated Plastic Flow Rule:

$$\dot{\boldsymbol{\epsilon}}_p = \dot{\lambda}\,\boldsymbol{s} \quad \text{where} \quad \boldsymbol{m} = \frac{\partial F}{\partial \boldsymbol{s}} = \boldsymbol{s}$$

Plastic Consistency Condition:

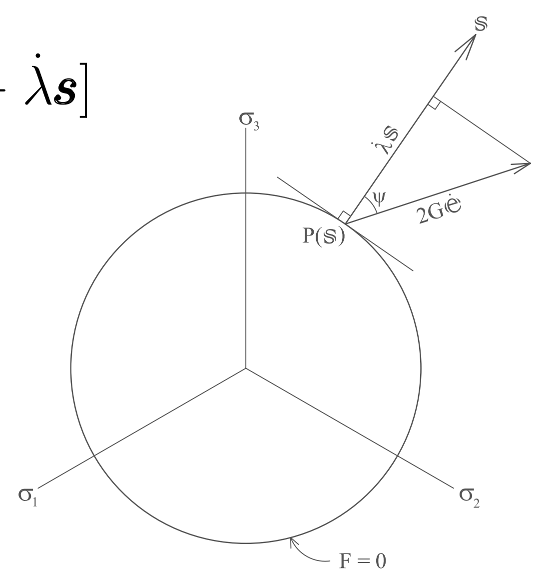
$$\dot{F} = \frac{\partial F}{\partial \boldsymbol{s}} : \dot{\boldsymbol{s}} = \boldsymbol{s} : \dot{\boldsymbol{s}} = 0$$

Deviatoric Stress Rate:

$$\dot{\boldsymbol{s}} = 2G\,[\dot{\boldsymbol{e}} - \dot{\boldsymbol{e}}_p] = 2G\,[\dot{\boldsymbol{e}} - \dot{\lambda}\boldsymbol{s}]$$

Plastic Multiplier:

$$\dot{\lambda} = \frac{\boldsymbol{s} : \dot{\boldsymbol{e}}}{\boldsymbol{s} : \boldsymbol{s}}$$



IDEAL J_2-ELASTOPLASTICITY II

Deviatoric Stress-Strain Relation:

$$\dot{\boldsymbol{s}} = 2G\,[\boldsymbol{I} - \frac{\boldsymbol{s} \otimes \boldsymbol{s}}{\boldsymbol{s} : \boldsymbol{s}}] : \dot{\boldsymbol{e}}$$

$$\dot{\boldsymbol{s}} = \boldsymbol{\mathcal{G}}_{ep} : \dot{\boldsymbol{e}} \quad \text{with} \quad \boldsymbol{\mathcal{G}}_{ep} = 2G\,[\boldsymbol{I} - \frac{\boldsymbol{s} \otimes \boldsymbol{s}}{\boldsymbol{s} : \boldsymbol{s}}]$$

Tangent Stiffness Operator:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{3}(tr\dot{\boldsymbol{\sigma}})\mathbf{1} + \dot{\boldsymbol{s}} = K(tr\dot{\boldsymbol{\epsilon}})\mathbf{1} + \boldsymbol{\mathcal{G}}_{ep} : \dot{\boldsymbol{e}}$$

$$\dot{\boldsymbol{\sigma}} = K(tr\dot{\boldsymbol{\epsilon}})\mathbf{1} + \boldsymbol{\mathcal{G}}_{ep} : [\dot{\boldsymbol{\epsilon}} - \frac{1}{3}(tr\dot{\boldsymbol{\epsilon}})\mathbf{1}]$$

Elastoplastic Tangent Operator

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{E}}_{ep} : \dot{\boldsymbol{\epsilon}} \quad \text{with} \quad \boldsymbol{\mathcal{E}}_{ep} = \Lambda\mathbf{1} \otimes \mathbf{1} + 2G\,[\boldsymbol{I} - \frac{\boldsymbol{s} \otimes \boldsymbol{s}}{\boldsymbol{s} : \boldsymbol{s}}]$$

Note: Elastoplastic constitutive structure similar to $K - G(\boldsymbol{e})$ model.

SIMPLE SHEAR EXAMPLE

von Mises vs parabolic Drucker-Prager: Response when $\dot{\gamma}_{12} > 0$

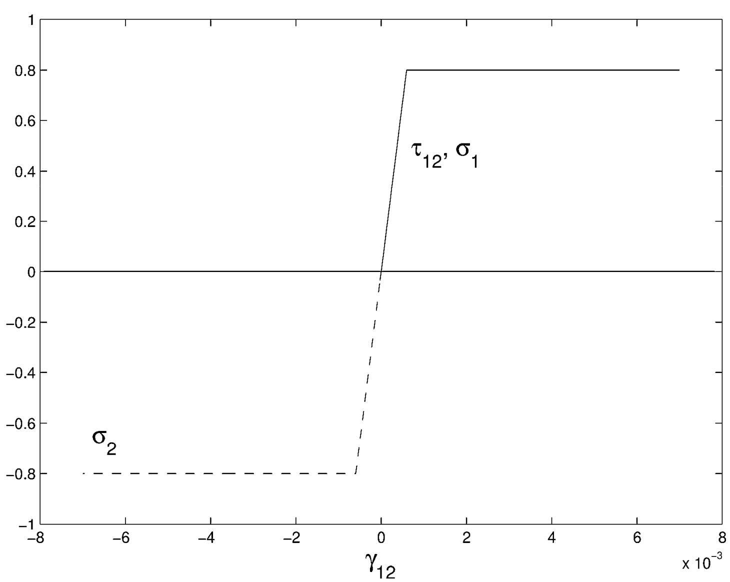
Parabolic Yield Function:

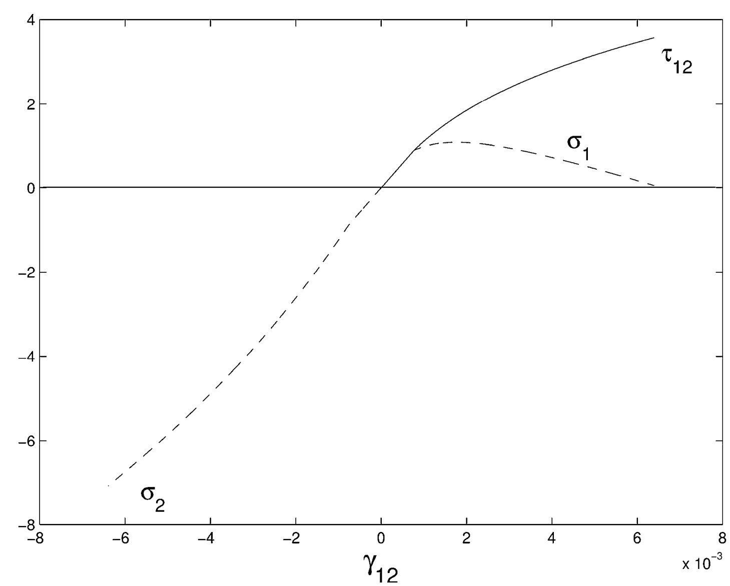
$$F(I_1, J_2) = J_2 + \alpha_F I_1 - \tau_y^2 = 0$$

Associated Flow Rule:

$$\dot{\boldsymbol{\epsilon}}_p = \dot{\lambda}[\boldsymbol{s} + \alpha_F \mathbf{1}]$$

Simple Shear: $\alpha_F = \frac{1}{3}[f_c' - f_t'] = 0$ for von Mises, while $\tau_Y^2 = \frac{1}{3} f_c' f_t' = \frac{1}{3}\sigma_Y^2$





GENERAL FORMULATION OF ELASTOPLASTIC BEHAVIOR I

Kinematic Setting: Decomposition of Total Deformation $\boldsymbol{\epsilon} = \frac{1}{2}[\nabla \boldsymbol{u} + \nabla^t \boldsymbol{u}]$,

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_e + \boldsymbol{\epsilon}_p$$

Elastic Behavior: Hyperelastic concept of free energy potential:

$$\Psi = \Psi(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}_p, \kappa)$$

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}_e} \qquad \text{and} \qquad \dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{E}} : [\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_p]$$

Plastic Yield Condition:

$$F(\boldsymbol{\sigma}, \kappa) = f(\boldsymbol{\sigma}) - r_y(\kappa) \leq 0 \qquad \text{with} \qquad \boldsymbol{n} = \frac{\partial F}{\partial \boldsymbol{\sigma}}$$

$f(\boldsymbol{\sigma})$ defines the internal stress demand and $r_y =$ the material resistance

$$r_y = \frac{\partial \Psi}{\partial \kappa} \qquad \text{and} \qquad \dot{r}_y = H_p \, \dot{\kappa}$$

Hardening modulus H_p characterizes the rate of yield resistance.

GENERAL ELASTOPLASTIC FORMULATION II

Plastic Flow Rule:

$$\dot{\boldsymbol{\epsilon}}_p = \dot{\lambda}\, \boldsymbol{m} \qquad \text{with} \qquad \boldsymbol{m} = \frac{\partial Q}{\partial \boldsymbol{\sigma}}$$

Associated flow when $\boldsymbol{m} \parallel \boldsymbol{n}$ (normality of plastic flow).

Plastic Consistency Condition: $\dot{F} = 0$

Consistency condition enforces the stress path to remain on the yield surface.

Kuhn–Tucker Condition of Plastic Loading:

$$F \leq 0 \qquad\qquad \dot{\lambda} \geq 0 \qquad\qquad F\,\dot{\lambda} = 0$$

Plastic Multiplier:

$$\dot{\lambda} = \frac{1}{h_p}\, \boldsymbol{n} : \boldsymbol{\mathcal{E}} : \dot{\boldsymbol{\epsilon}} \quad \text{with} \quad h_p = H_p + \boldsymbol{n} : \boldsymbol{\mathcal{E}} : \boldsymbol{m}$$

GENERAL ELASTOPLASTIC FORMULATION III

Elastoplastic Stiffness Relation:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{E}} : [\dot{\boldsymbol{\epsilon}} - \dot{\lambda}\boldsymbol{m}] = \boldsymbol{\mathcal{E}} : [\dot{\boldsymbol{\epsilon}} - \boldsymbol{m}\frac{\boldsymbol{n} : \boldsymbol{\mathcal{E}} : \dot{\boldsymbol{\epsilon}}}{H_p + \boldsymbol{n} : \boldsymbol{\mathcal{E}} : \boldsymbol{m}}]$$

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{E}}_{ep} : \dot{\boldsymbol{\epsilon}}$$

Note #1: Plastic stiffness forms rank–one (two) update of the elastic material operator

$$\boldsymbol{\mathcal{E}}_{ep} = \boldsymbol{\mathcal{E}} - \frac{1}{h_p}\boldsymbol{\mathcal{E}} : \boldsymbol{m} \otimes \boldsymbol{n} : \boldsymbol{\mathcal{E}}$$

where $h_p = H_p + \boldsymbol{n} : \boldsymbol{\mathcal{E}} : \boldsymbol{m}$.

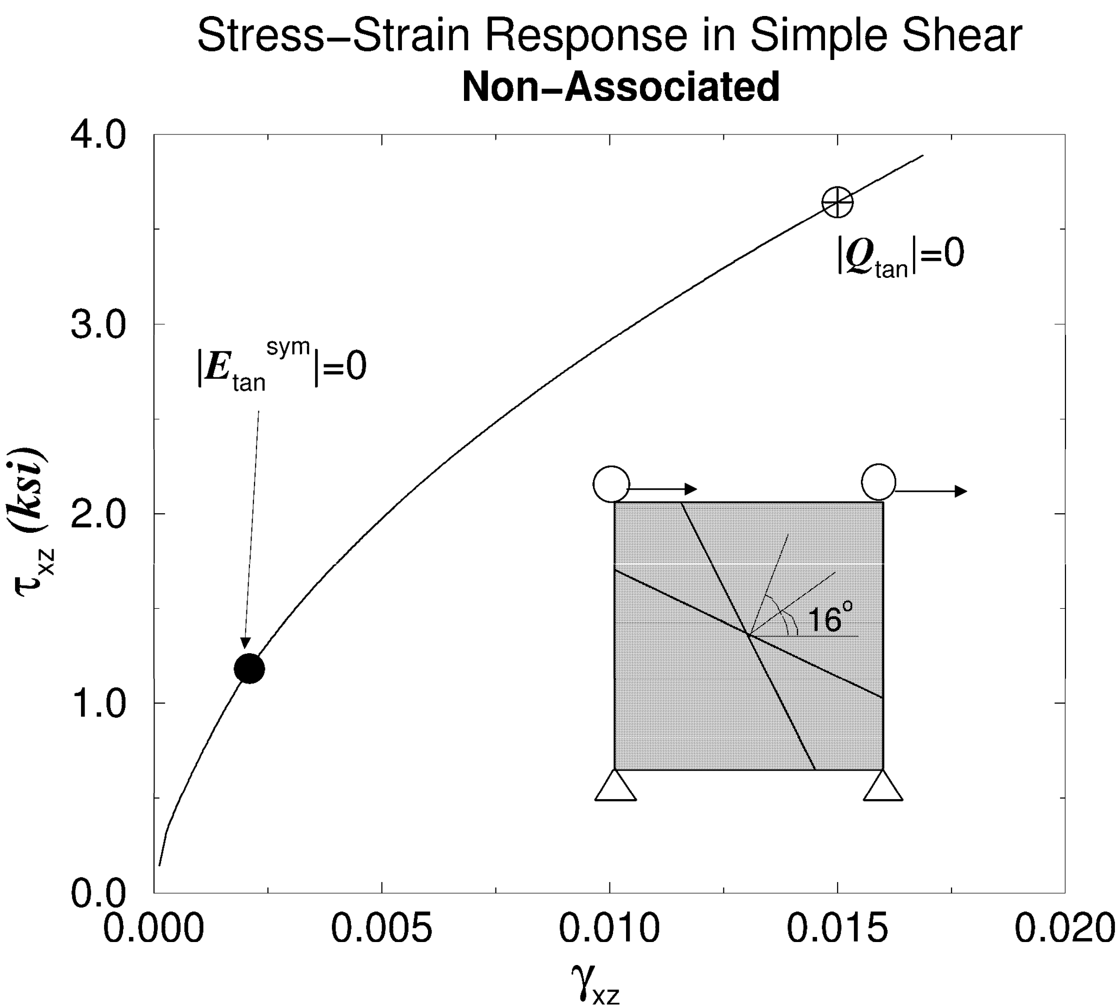
Note #2: $h_p = 0$ when softening modulus reaches $H_p^{crit} = -\boldsymbol{n} : \boldsymbol{\mathcal{E}} : \boldsymbol{m}$.

Note #3: Loss of symmetry, $\boldsymbol{\mathcal{E}}_{ep} \neq \boldsymbol{\mathcal{E}}_{ep}^t$ when $\boldsymbol{n} \neq \boldsymbol{m}$ for non-associated flow.

SIMPLE SHEAR RESPONSE

Three Invariant Elastoplastic Concrete Model: Kang and Willam [1999]

Effect of Confinement under Strain Control



CONCLUDING REMARKS

Main Lessons from Class # 3:

Flow Theory of Plasticity:

introduces path-dependence, irreversibility and energy dissipation

Canonical Form of J_2 Elastoplasticity:

Decouples volumetric-deviatoric behavior, see $K - G(\boldsymbol{e})$ *model*

Volumetric-Deviatoric Coupling:

Two and three invariant elastoplastic models - Isotropic hardening/softening compares to rotating crack approach (no crack/slip memory)

Smeared Cracking in Form of Plastic Softening of Major Strain Component

$\boldsymbol{\epsilon} = -\frac{\nu}{E}(tr\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2G}\boldsymbol{\sigma} + C_N \sigma_1 [\boldsymbol{e}_1 \otimes \boldsymbol{e}_1]$

Softening Rankine plasticity is equivalent to rotating crack formulation using elastic damage.