Class Meeting #4: Failure Analysis at Constitutive Level

Continuous vs Discontinuous Failure: Continuum ⇒ Discontinuum

Loss of Material Stability/Uniqueness: \( d^2W = 0 \) vs \( \text{det} \mathbf{E} = 0 \)

Loss of Ellipticity/Hyperbolicity: Localization Analysis

Geometric Localization Criterion: Elliptic Localization Envelope
MOHR FAILURE ENVELOPE

Geometric Concept: \( F(\sigma, \tau) = f(\sigma) - r_y = 0 \)

O. Mohr [1900]: Critical Mohr circle contacts failure envelope

\[
f(\sigma) = \frac{1}{2} [\sigma_1 - \sigma_3]
\]

Is there a universal ‘strength criterion’ for brittle-ductile failure
DEGRADATION OF KINEMATIC CONTINUITY

Transition of Continuum into Discontinuum

1. Diffuse Failure: continuous velocities and velocity gradients

2. Localized Failure: formation of weak discontinuities

3. Discrete Failure: formation of strong discontinuities
BOND AT BIMATERIAL INTERFACE

Perfect Contact:

\[ [u_N] = u^b_N - u^m_N = 0 \quad \text{and} \quad [t_N] = t^b_N - t^m_N = 0 \]

Weak Discontinuities: all strain components exhibit jumps across interface except for \( \epsilon^b_{TT} = \epsilon^m_{TT} \) restraint.

Note: Jump of tangential normal stress, \( \sigma^b_{TT} \neq \sigma^m_{TT} \).

Imperfect Contact:

\[ [u_N] = u^b_N - u^m_N \neq 0 \quad \text{whereas} \quad [t_N] = t^b_N - t^m_N = 0 \]

Strong Discontinuities: all displacement components exhibit jumps across interface.

Note: FE Displacement method enforces traction continuity in ‘weak’ sense only, hence \( [t_N] \neq 0 \).
MATERIAL STABILITY

Second Order Work Density Functional:

\[
d^2W = \frac{1}{2} \dot{\sigma} : \dot{\epsilon} = \frac{1}{2} \dot{\epsilon} : \mathcal{E}_{tan} : \dot{\epsilon} > 0, \forall \dot{\epsilon} \neq 0
\]

For non-associated plasticity:

\[
d^2W = \frac{1}{2} \dot{\epsilon} : \mathcal{E} : \dot{\epsilon} - \frac{1}{4h_p} \dot{\epsilon} : \left[ \bar{m} \otimes \bar{n} + \bar{n} \otimes \bar{m} \right] : \dot{\epsilon}
\]

Note: The energy functional uses only the symmetric tangent operator.

Bromwich Eigenvalue Bounds:

\[
\lambda_{min}(\mathcal{E}_{ep}^{sym}) \leq \Re(\lambda_{min}(\mathcal{E}_{ep})) \leq \lambda_{max}(\mathcal{E}^{sym})
\]

Material instability coincides with loss of positive definiteness of the symmetric operator: \(\lambda_{min}(\mathcal{E}_{ep}^{sym}) = 0\)

Critical Hardening Modulus, [Maier & Hueckl, 1979]:

\[
H_p^{stabil} = \frac{1}{2} \left[ \sqrt{(n : \mathcal{E} : n)(m : \mathcal{E} : m)} - n : \mathcal{E} : m \right]
\]
MATERIAL UNIQUENESS

Loss of Material Uniqueness: $\dot{\sigma} = \mathcal{E}_{ep} : \dot{\epsilon} = 0$

Indicates stationary stress state at limit point. Loss of uniqueness is synonymous with the formation of a singular tangent operator.

$$\det \mathcal{E}_{ep} \stackrel{!}{=} 0 \quad \rightarrow \quad \lambda_{\text{min}}(\mathcal{E}_{ep}) = 0$$

The plastic operator is a rank-one update of the positive elasticity tensor,

$$\mathcal{E}_{ep} = \mathcal{E} - \frac{1}{h_p} \bar{m} \otimes \bar{n}$$

Pre-conditioning

$$\mathcal{E}^{-1} : \mathcal{E}^{ep} = \mathbf{I} - \mathcal{E}^{-1} : \frac{\bar{m} \otimes \bar{n}}{h_p}$$

Critical eigenvalue $\lambda_{\text{min}}$ measures uniqueness by scalar damage variable $d_\mathcal{E}$,

$$\lambda_{\text{min}} \left( \mathcal{E}^{-1} : \mathcal{E}^{ep} \right) = 1 - d_\mathcal{E} \quad \text{with} \quad d_\mathcal{E} := \frac{n : \mathcal{E} : m}{H_p + n : \mathcal{E} : m}$$

Note: $H_p^{\text{limit}} = 0$ corresponds to $1 - d_\mathcal{E} \stackrel{!}{=} 0$ or to $d_\mathcal{E} = 1$. 
LOCALIZATION ANALYSIS

Kinematic Compatibility across Discontinuity Surface:

\[
|\nabla \dot{u}| = M \otimes N \quad \rightarrow \quad |\dot{\varepsilon}| = \frac{1}{2} \left[ M \otimes N + N \otimes M \right]
\]

Traction Equilibrium: Cauchy’s Theorem

\[
[|\dot{t}|] = \dot{t}^+ - \dot{t}^- = 0
\]

\[
[|\dot{t}|] = N \cdot [|\dot{\sigma}|] = N \cdot [|\varepsilon_{\text{tan}} : \dot{\varepsilon}|] = 0
\]

Assuming \(|\varepsilon_{\text{tan}}| = \varepsilon_{\text{tan}}^+ - \varepsilon_{\text{tan}}^- = 0\)

Continuous Material Bifurcation:

\[
Q_{\text{tan}} \cdot M = 0 \quad \text{with} \quad Q_{\text{tan}} = N \cdot \varepsilon_{\text{tan}} \cdot N
\]

\(Q_{\text{tan}}\) is the tangential localization tensor with Localization Criterion:

\[
\det Q_{\text{tan}} \neq 0 \quad \rightarrow \quad \lambda_{\text{min}}(Q_{\text{tan}}) = 0
\]
ELASTOPLASTIC LOCALIZATION CONDITION

Elastoplastic Bifurcation Condition:

\[ \det(Q_{ep}) = \det(N \cdot [E_o - \frac{1}{h_p}E_o : m \otimes n : E_o] \cdot N) = 0 \]

Rank–one Update Format of Elastoplastic Localization Tensor:

\[ Q_{ep} = Q_0 - \frac{1}{h_p}e_m \otimes e_n \]

where

\[ e_m = N \cdot E : m \]
\[ e_n = n : E \cdot N \]

Discontinuous Failure Mode: \[ [\dot{\varepsilon}] = \frac{1}{2} \left[ M \otimes N + N \otimes M \right] \]

Failure orientation depends on: \( m = \frac{\partial Q}{\partial \sigma} \) and \( n = \frac{\partial F}{\partial \sigma} \)
LOCALIZATION ELLIPSE

Scalar Form of Localization Condition:

\[ H_p^{loc} + n : \mathcal{E} : m = e_n \cdot Q^{-1} \cdot e_m \]

Geometric Envelope Condition:

\[ \frac{(\sigma - \sigma_0)^2}{A^2} + \frac{\tau^2}{B^2} = 1 \]
LOCALIZED FAILURE MODE

Parabolic Drucker-Prager:

\[ F = J_2 + \alpha_F I_1 - \beta_F \quad \text{and} \quad Q = J_2 + \alpha_Q I_1 - \beta_Q \]

Character of Jump: \[ [\ddot{\epsilon}] = \frac{1}{2} \begin{bmatrix} M \otimes N + N \otimes M \end{bmatrix} \]

Half Axes of Localization Ellipse:

\[ A^2 = \frac{2(1-\nu)}{1-2\nu} B^2 \quad \text{and} \quad B^2 = \frac{1}{4G} H_{ploc}^2 + J_2 + \frac{1-\nu}{8(1-2\nu)}(\alpha_F + \alpha_Q)^2 + \frac{1+2\nu}{1-2\nu} \alpha_F \alpha_Q \]

Critical Normal Vector of Failure Plane \( N \) w/r to major principal \( e_1 \)-axis:

\[ \tan^2 \theta_{cr} = \frac{r - [(1 - 2\nu)(\sigma_c - \frac{1}{3} I_1) + \frac{1}{2}(1 - \nu)(\alpha_F + \alpha_Q)]}{r + [(1 - 2\nu)(\sigma_c - \frac{1}{3} I_1) + \frac{1}{2}(1 - \nu)(\alpha_F + \alpha_Q)]} \]

Critical Hardening Modulus:

\[ H_{pcr} = 4G \left\{ r^2 + (1-2\nu) \left[ \sigma_c - \frac{1}{3} I_1 + \frac{(1 - \nu)(\alpha_F + \alpha_Q)}{2(1 - 2\nu)} \right]^2 - J_2 - \frac{(1 - \nu)(\alpha_F + \alpha_Q)^2}{8(1 - 2\nu)} - \frac{(1 + 2\nu)}{(1 - 2\nu)} \right\} \]
MODEL PROBLEM OF SIMPLE SHEAR: $\dot{\gamma}_{12} > 0$

Non-Associated Parabolic Drucker-Prager Model: $\alpha_Q = 0$

Failure Mode (angle $\theta^{\text{crit}}$) and Contrast Strength Ratios $f'_c : f'_t$

<table>
<thead>
<tr>
<th>Confinement</th>
<th>$\theta^{\text{crit}}$</th>
</tr>
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<tbody>
<tr>
<td>1:1</td>
<td>45.00</td>
</tr>
<tr>
<td>3:1</td>
<td>35.26</td>
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<tr>
<td>5:1</td>
<td>29.45</td>
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<tr>
<td>8:1</td>
<td>22.20</td>
</tr>
<tr>
<td>12:1</td>
<td>11.78</td>
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</tbody>
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Critical Failure Angle of Parabolic Drucker-Prager Model – Simple Shear Problem

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MODEL PROBLEM OF SIMPLE SHEAR: $\dot{\gamma}_{12} > 0$

Geometric Localization Analysis: $f'_c : f'_t = 1 : 1$, von Mises

Pressure-Insensitive Failure: $\theta^{\text{crit}} = 45^0$ and $\theta^{\text{crit}} = 135^0$ shear failure mode II.
MODEL PROBLEM OF SIMPLE SHEAR: $\dot{\gamma}_{12} > 0$

Geometric Localization Analysis: $f'_c : f'_t = 3 : 1$

Pressure-Sensitive Failure: $\theta = 33.211^0$ and $\theta = 146.79^0$ indicate mixed shear-compression failure.
MODEL PROBLEM OF SIMPLE SHEAR: $\dot{\gamma}_{12} > 0$

Geometric Localization Analysis: $f_c' : f_t' = 12 : 1$

Highly Pressure-Sensitive Failure: $\theta^{\text{crit}} = 0^0$ indicates brittle failure mode I.
MODE I: SPLITTING TENSION

Critical Localization Condition: Mode I when $N||M$ and $\theta^{cr} = 0$

Associated Flow: $\alpha_F = \alpha_Q = 0.25$
Non-Associated Flow: $\alpha_F = 1.167$, $\alpha_Q = -0.667$
MODE I: SPLITTING COMPRESSION

Critical Localization Condition: Mode I when $\mathbf{N} \parallel \mathbf{M}$ and $\theta_{cr} = 0$

Associated Flow: $\alpha_F = \alpha_Q = 3.0$
Non-Associated Flow: $\alpha_F = 1.167, \alpha_Q = 4.833$
Critical Localization Condition: Compaction Band

Associated Flow for Compaction Band: \( \alpha_F = \alpha_Q = -2.0 \)
CONCLUDING REMARKS

Main Lessons from Class # 4:

Diffuse Failure:
- *Loss of Stability and Loss of Uniqueness*

Localized Failure:
- *Loss of Ellipticity and Hyperbolicity*

Volumetric-Deviatoric Coupling:
- *Simple Shear Test Exhibits Confinement Effects*

Compression Failure of Brittle Materials:
- *Splitting Compression Depends on Confinement*