MODELING OF CONCRETE MATERIALS AND STRUCTURES

Kaspar Willam
University of Colorado at Boulder

Class Meeting #5: Integration of Constitutive Equations

Structural Equilibrium:
Incremental Tangent Stiffness and Residual Force Iteration

Radial Return Method:
Elastic Predictor-Plastic Corrector Strategy

Algorithmic Tangent Operator:
Consistent Tangent with Backward Constitutive Integration

STRUCTURAL EQUILIBRIUM

Out-of-Balance Force Calculation:

Out-of Balance Residual Forces: $\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{F}_{int} - \boldsymbol{F}_{ext} \rightarrow \mathbf{0}$

Internal Forces: $\boldsymbol{F}_{int} = \sum_e \int_V \boldsymbol{B}^t \boldsymbol{\sigma} dV$

External Forces: $\boldsymbol{F}_{ext} = \sum_e \int_V \boldsymbol{N}^t \boldsymbol{b} dV + \sum_e \int_S \boldsymbol{N}^t \boldsymbol{t} dS$

Rate of Equilibrium:

$$\sum_e \int_V \boldsymbol{B}^t \dot{\boldsymbol{\sigma}} dV = \sum_e \int_V \boldsymbol{N}^t \dot{\boldsymbol{b}} dV + \sum_e \int_S \boldsymbol{N}^t \dot{\boldsymbol{t}} dS$$

1. Incremental Methods of Numerical Integration: Path-following continuation strategies to advance solution within $\Delta t = t_{n+1} - t_n$

(a) Explicit Euler Forward Approach: Forward tangent stiffness strategy (should include out-of-balance equilibrium corrections to control drift).

(b) Implicit Euler Backward Approach: Backward tangent stiffness (requires iteration for calculating tangent stiffness at end of increment; Heun's method at midstep, and Runge-Kutta h/o methods at intermediate stages).

INCREMENTAL SOLVERS

Euler Forward Integration: Classical Tangent stiffness approach

Internal forces: $\dot{\boldsymbol{F}}_{ext} = \sum_e \int_V \boldsymbol{B}^t \dot{\boldsymbol{\sigma}} dV$

Tangential Material Law: $\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{E}}_{tan} \dot{\boldsymbol{\epsilon}}$

Tangential Stiffness Relationship:

$$\boldsymbol{K}_{tan} \dot{\boldsymbol{u}} = \dot{\boldsymbol{F}} \quad \text{where} \quad \boldsymbol{K}_{tan} = \sum_e \int_V \boldsymbol{B}^t \boldsymbol{E}_{tan} \boldsymbol{B} dV$$

Incremental Format: $\int_{u_n}^{u_{n+1}} \boldsymbol{K}_{tan} d\boldsymbol{u} = \boldsymbol{F}_{n+1} - \boldsymbol{F}_n$

Euler Forward Integration: $\boldsymbol{K}_{tan}^n = \sum_e \int_V \boldsymbol{B}^t \boldsymbol{E}_{tan}^n \boldsymbol{B} dV$

$$\boldsymbol{K}_{tan}^n \Delta \boldsymbol{u} = \Delta \boldsymbol{F} \quad \text{where} \quad \boldsymbol{E}_{tan}^n = \boldsymbol{E}_{tan}(t_n)$$

Note: Uncontrolled drift of response path if no equilibrium corrections are included at each load step.

Picard direct substitution iteration vs Newton-Raphson iteration within $\Delta t = t_{n+1} - t_n$

Robustness Issues: Range and Rate of Convergence?

Newton-Raphson Residual Force Iteration: $\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{0}$

Truncated Taylor Series Expansion of the residual $\boldsymbol{R}$ around $\boldsymbol{u}^{i-1}$ yields

$$\boldsymbol{R}(\boldsymbol{u})^i = \boldsymbol{R}(\boldsymbol{u})^{i-1} + \left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{u}}\right)^{i-1}[\boldsymbol{u}^i - \boldsymbol{u}^{i-1}]$$

Letting $\boldsymbol{R}(\boldsymbol{u}^i) = \boldsymbol{0}$ solve

$$\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{u}}\right)^{i-1}[\boldsymbol{u}^i - \boldsymbol{u}^{i-1}] = -\boldsymbol{R}(\boldsymbol{u})^{i-1}$$

NEWTON-RAPHSON EQUILIBRIUM ITERATION

Assuming conservative external forces: $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{u}} = \sum_e \int_V \boldsymbol{B}^t \frac{d\boldsymbol{\sigma}}{d\boldsymbol{u}} dV$

Chain rule of differentiation leads to

$d\boldsymbol{\sigma} = \boldsymbol{E}_{tan} d\boldsymbol{\epsilon} = \boldsymbol{E}_{tan} \boldsymbol{B} d\boldsymbol{u}$ such that $\frac{d\boldsymbol{\sigma}}{d\boldsymbol{u}} = \frac{d\boldsymbol{\sigma}}{d\boldsymbol{\epsilon}} \frac{d\boldsymbol{\epsilon}}{d\boldsymbol{u}} = \boldsymbol{E}_{tan} \boldsymbol{B}$.

Tangent stiffness matrix provides Jacobian of N-R residual iteration,

$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{u}} = \boldsymbol{K}_{tan}$ where $\boldsymbol{K}_{tan} = \int_V \boldsymbol{B}^t \boldsymbol{E}_{tan} \boldsymbol{B} dV$

Newton-Raphson Equilibrium Iteration:

$$\boldsymbol{K}_{tan}^{i-1} [\boldsymbol{u}^i - \boldsymbol{u}^{i-1}] = -\boldsymbol{R}(\boldsymbol{u})^{i-1}$$

For i=1, the starting conditions for the first iteration cycle are,

$$\boldsymbol{K}_{tan}^{n} [\boldsymbol{u}^1 - \boldsymbol{u}^{n}] = \boldsymbol{F}^{n+1} - \int_V \boldsymbol{B}^t \boldsymbol{\sigma}^n$$

First iteration cycle coincides with Euler forward step, whereby each equilibrium iteration requires updating the tangential stiffness matrix.

Note: Difficulties near limit point when $\det \boldsymbol{K}_{tan} \rightarrow 0$

RADIAL RETURN METHOD OF J_2-PLASTICITY I

Mises Yield Function:

$$F(\boldsymbol{s}) = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s} - \frac{1}{3}\sigma_Y^2 = 0$$

Associated Plastic Flow Rule:

$$\dot{\boldsymbol{\epsilon}}_p = \dot{\lambda}\,\boldsymbol{s} \quad \text{where} \quad \boldsymbol{m} = \frac{\partial F}{\partial \boldsymbol{s}} = \boldsymbol{s}$$

Plastic Consistency Condition:

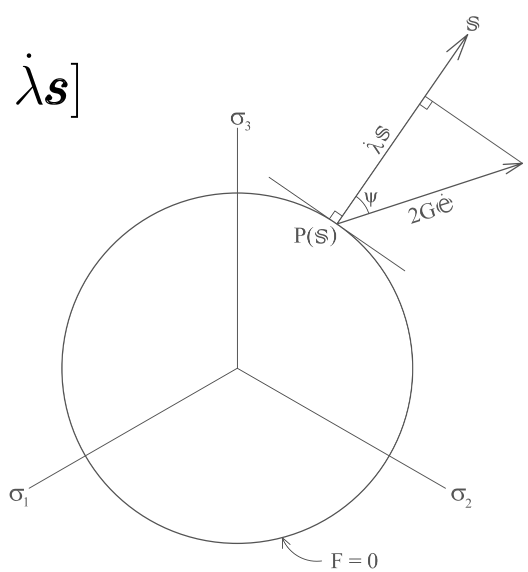
$$\dot{F} = \frac{\partial F}{\partial \boldsymbol{s}} : \dot{\boldsymbol{s}} = \boldsymbol{s} : \dot{\boldsymbol{s}} = 0$$

Deviatoric Stress Rate:

$$\dot{\boldsymbol{s}} = 2G\,[\dot{\boldsymbol{e}} - \dot{\boldsymbol{e}}_p] = 2G\,[\dot{\boldsymbol{e}} - \dot{\lambda}\boldsymbol{s}]$$

Plastic Multiplier:

$$\dot{\lambda} = \frac{\boldsymbol{s} : \dot{\boldsymbol{e}}}{\boldsymbol{s} : \boldsymbol{s}}$$



RADIAL RETURN METHOD OF J_2-PLASTICITY II

Incremental Format:

$$\Delta \boldsymbol{s} = 2G\left[\Delta \boldsymbol{e} - \Delta\lambda \boldsymbol{s}\right]$$

Elastic Predictor-Plastic Corrector Split:

(a) Elastic Predictor: $\boldsymbol{s}_{trial} = \boldsymbol{s}_n + 2G\Delta\boldsymbol{e}$
(b) Plastic Corrector: $\boldsymbol{s}_{n+1} = \boldsymbol{s}_{trial} - 2G\Delta\lambda\boldsymbol{s}_{trial} = [1 - 2G\Delta\lambda]\boldsymbol{s}_{trial}$

"Full" Consistency: $F_{n+1} = \frac{1}{2}\boldsymbol{s}_{n+1} : \boldsymbol{s}_{n+1} - \frac{1}{3}\sigma_Y^2 = 0$

Quadratic equation for computing plastic multiplier $\Delta\lambda_1 =?$ and $\Delta\lambda_2 =?$

$$\frac{1}{2}[\boldsymbol{s}_{trial} - 2G\Delta\lambda\boldsymbol{s}_{trial}] : [\boldsymbol{s}_{trial} - 2G\Delta\lambda\boldsymbol{s}_{trial}] = \frac{1}{3}\sigma_Y^2$$

Plastic Multiplier: $\Delta\lambda_{min} = \frac{1}{2G}[1 - \sqrt{\frac{2}{3}}\frac{\sigma_Y}{\sqrt{\sigma_{trial}:\sigma_{trial}}}]$

"Radial Return": represents closest point projection of the trial stress state onto the yield surface. Final stress state is the scaled-back trial stress,

$$\boldsymbol{s}_{n+1} = \sqrt{\frac{2}{3}}\frac{\sigma_Y}{\sqrt{\boldsymbol{s}_{trial} : \boldsymbol{s}_{trial}}}\boldsymbol{s}_{trial}$$

GENERAL FORMAT OF PLASTIC RETURN METHOD

Incremental Format:

$$\Delta\boldsymbol{\sigma} = \boldsymbol{E} : [\Delta\boldsymbol{\epsilon} - \Delta\lambda\boldsymbol{m}]$$

Elastic Predictor-Plastic Corrector Split:

(a) Elastic Predictor: $\boldsymbol{\sigma}_{trial} = \boldsymbol{\sigma}_n + \boldsymbol{E} : \Delta\boldsymbol{\epsilon}$
(b) Plastic Corrector: $\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{trial} - \Delta\lambda\boldsymbol{E} : \boldsymbol{m}$

"Full" Consistency: $F_{n+1} = F(\boldsymbol{\sigma}_n + \boldsymbol{E} : \Delta\boldsymbol{\epsilon} - \Delta\lambda\boldsymbol{E} : \boldsymbol{m}) = 0$

(i) Explicit Format: $\boldsymbol{m} = \boldsymbol{m}_n$ (or evaluate $\boldsymbol{m}$ at $\boldsymbol{m} = \boldsymbol{m}_c$ or $\boldsymbol{m} = \boldsymbol{m}_{trial}$)

Use N-R for solving $\Delta\lambda =?$ for a given direction of plastic return e.g. $\boldsymbol{m} = \boldsymbol{m}_n$.

(i) Implicit Format: $\boldsymbol{m} = \boldsymbol{m}_{n+\alpha}$ where $0 < \alpha \leq 1$ ($\boldsymbol{m} = \boldsymbol{m}_{n+1}$ for BEM).

$$F_{n+1} = F(\boldsymbol{\sigma}_n + \boldsymbol{E} : \Delta\boldsymbol{\epsilon} - \Delta\lambda\boldsymbol{E} : \boldsymbol{m}_{n+\alpha}) = 0$$

Use N-R for solving $\Delta\lambda =?$ in addition to unknown $\boldsymbol{m} = \boldsymbol{m}_{n+\alpha}$

ALGORITHMIC TANGENT STIFFNESS

Consistent Tangent vs Continuum Tangent:

Uniaxial Example:

$$\dot{\sigma} = E_{tan}\dot{\epsilon} \quad \text{where} \quad E_{tan} = E_0[1 - \frac{\sigma}{\sigma_0}] \quad \text{hence} \quad \sigma = \sigma_0[1 - e^{-\frac{E_0}{\sigma_0}\epsilon}]$$

Fully Implicit Euler Backward Integration: $E_{tan} = E_{tan}^{n+1}$ in $\Delta t = t_{n+1} - t_n$,

$$\sigma^{n+1} = \sigma^n + E_0[1 - \frac{\sigma^{n+1}}{\sigma_0}][\epsilon^{n+1} - \epsilon^n]$$

Algorithmic Tangent Stiffness: Relates $d\epsilon^{n+1}$ to $d\sigma^{n+1}$ at t_{n+1}

$$d\sigma^{n+1} = E_{tan}^{alg}\, d\epsilon^{n+1} \quad \text{where} \quad E_{tan}^{alg} = \frac{E_0[1 - \frac{\sigma^{n+1}}{\sigma_0}]}{1 + \frac{E_0}{\sigma_0}\Delta\epsilon}$$

Ratio of Tangent Stiffness Properties: $\frac{E_{tan}^{alg}}{E_{tan}} = \frac{1}{1+\frac{E_0}{\sigma_0}\Delta\epsilon} \sim 0.7$

ALGORITHMIC TANGENT STIFFNESS OF J_2-PLASTICITY

Incremental Form of Elastic-Plastic Split:

$$\Delta \boldsymbol{s} = 2G\,[\Delta \boldsymbol{e} - \Delta\lambda \boldsymbol{s}]$$

Fully Implicit Euler Backward Integration: for $\Delta t = t_{n+1} - t_n$,

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}_n + 2G[\boldsymbol{e}_{n+1} - \boldsymbol{e}_n] - \Delta\lambda 2G \boldsymbol{s}_{n+1}$$

Relating $d\boldsymbol{s}^{n+1}$ to $d\boldsymbol{e}^{n+1}$ at t_{n+1}, differentiation yields,

$$d\boldsymbol{s}_{n+1} = 2G\,d\boldsymbol{e}_{n+1} - d\Delta\lambda 2G \boldsymbol{s}_{n+1} - \Delta\lambda 2G d\boldsymbol{s}_{n+1}$$

Algorithmic Tangent Stiffness Relationship:

$$d\boldsymbol{s}_{n+1} = \frac{2G}{1+\Delta\lambda 2G}[\boldsymbol{I} - \frac{\boldsymbol{s}_{n+1} \otimes \boldsymbol{s}_{n+1}}{\boldsymbol{s}_{n+1} : \boldsymbol{s}_{n+1}}] : d\boldsymbol{e}_{n+1}$$

Ratio of Tangent Stiffness Properties: $\frac{G^{alg}_{tan}}{G^{cont}_{tan}} = \frac{1}{1+\Delta\lambda 2G}$ when $||\boldsymbol{s}_{n+1}|| \sim ||\boldsymbol{s}_n||$.

CONCLUDING REMARKS

Main Lessons from Class # 5:

Nonlinear Solvers:

Forward Euler method introduces drift from true response path. Newton-Raphson Iteration exhibits convergence difficulties when $\boldsymbol{K}_{tan} \rightarrow 0$ *(ill-conditioning).*

CPPM for Computational Plasticity:

Analytical Radial Return solution available for J_2*-plasticity and Drucker-Prager. Generalization leads to explicit and implicit plastic return strategies which are nowadays combined with the incremental hardening and incremental stress residuals in a monolithic Newton strategy.*

Algorithmic vs Continuum Tangent:

For quadratic convergence tangent operator must be consistent with integration of constitutive equations. The algorithmic tangent compares to secant stiffness in increments which are truly finite.