MODELING OF CONCRETE MATERIALS AND STRUCTURES

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Class Meeting #5: Integration of Constitutive Equations

Structural Equilibrium:

*Incremental Tangent Stiffness and Residual Force Iteration*

Radial Return Method:

*Elastic Predictor-Plastic Corrector Strategy*

Algorithmic Tangent Operator:

*Consistent Tangent with Backward Constitutive Integration*
STRUCTURAL EQUILIBRIUM

Out-of-Balance Force Calculation:

Out-of Balance Residual Forces: \( R(u) = F_{int} - F_{ext} \to 0 \)

Internal Forces: \( F_{int} = \sum_e \int_V B^t \sigma dV \)

External Forces: \( F_{ext} = \sum_e \int_V N^t b dV + \sum_e \int_S N^t t dS \)

Rate of Equilibrium:

\[
\sum_e \int_V B^t \dot{\sigma} dV = \sum_e \int_V N^t \dot{b} dV + \sum_e \int_S N^t \dot{t} dS
\]

1. Incremental Methods of Numerical Integration: Path-following continuation strategies to advance solution within \( \Delta t = t_{n+1} - t_n \)

(a) Explicit Euler Forward Approach: Forward tangent stiffness strategy (should include out-of-balance equilibrium corrections to control drift).

(b) Implicit Euler Backward Approach: Backward tangent stiffness (requires iteration for calculating tangent stiffness at end of increment; Heun’s method at midstep, and Runge-Kutta h/o methods at intermediate stages).
INCREMENTAL SOLVERS

Euler Forward Integration: Classical Tangent stiffness approach

Internal forces: \( \dot{F}_{\text{ext}} = \sum_e \int_V B^t \dot{\sigma} dV \)

Tangential Material Law: \( \dot{\sigma} = \dot{E}_{\text{tan}} \dot{\epsilon} \)

Tangential Stiffness Relationship:

\[
K_{\text{tan}} \dot{u} = \dot{F} \quad \text{where} \quad K_{\text{tan}} = \sum_e \int_V B^t E_{\text{tan}} B dV
\]

Incremental Format:

\[
\int_{u_n}^{u_{n+1}} K_{\text{tan}} du = F_{n+1} - F_n
\]

Euler Forward Integration:

\[
K_{\text{tan}}^n \Delta u = \Delta F \quad \text{where} \quad E_{\text{tan}}^n = E_{\text{tan}}(t_n)
\]

Note: Uncontrolled drift of response path if no equilibrium corrections are included at each load step.
2. ITERATIVE SOLVERS

Picard direct substitution iteration vs Newton-Raphson iteration within
\[ \Delta t = t_{n+1} - t_n \]

Robustness Issues: Range and Rate of Convergence?

Newton-Raphson Residual Force Iteration:

\[ R(u) = 0 \]

Truncated Taylor Series Expansion of the residual \( R \) around \( u^{i-1} \) yields

\[ R(u)^i = R(u)^{i-1} + \left( \frac{\partial R}{\partial u} \right)^{i-1}[u^i - u^{i-1}] \]

Letting \( R(u^i) = 0 \) solve

\[ \left( \frac{\partial R}{\partial u} \right)^{i-1}[u^i - u^{i-1}] = -R(u)^{i-1} \]
Newton-Raphson Equilibrium Iteration

Assuming conservative external forces: \( \frac{\partial R}{\partial u} = \sum e \int_V B^t d\sigma dV \)

Chain rule of differentiation leads to

\[ d\sigma = E_{\tan} d\epsilon = E_{\tan} B d\mathbf{u} \]

such that \( \frac{d\sigma}{du} = \frac{d\sigma}{d\epsilon} \frac{d\epsilon}{du} = E_{\tan} B \).

Tangent stiffness matrix provides Jacobian of N-R residual iteration,

\[ \frac{\partial R}{\partial u} = K_{\tan} \] where \( K_{\tan} = \int_V B^t E_{\tan} B dV \)

Newton-Raphson Equilibrium Iteration:

\[ K_{\tan}^{i-1} \left[ \mathbf{u}^i - \mathbf{u}^{i-1} \right] = -R(\mathbf{u})^{i-1} \]

For \( i=1 \), the starting conditions for the first iteration cycle are,

\[ K_{\tan}^n \left[ \mathbf{u}^1 - \mathbf{u}^n \right] = F^{n+1} - \int_V B^t \sigma^n \]

First iteration cycle coincides with Euler forward step, whereby each equilibrium iteration requires updating the tangential stiffness matrix.

Note: Difficulties near limit point when \( \det K_{\tan} \rightarrow 0 \)
RADIAL RETURN METHOD OF $J_2$-PLASTICITY I

Mises Yield Function:

$$F(s) = \frac{1}{2} s : s - \frac{1}{3} \sigma_Y^2 = 0$$

Associated Plastic Flow Rule:

$$\dot{\epsilon}_p = \dot{\lambda} s \quad \text{where} \quad m = \frac{\partial F}{\partial s} = s$$

Plastic Consistency Condition:

$$\dot{F} = \frac{\partial F}{\partial s} : \dot{s} = s : \dot{s} = 0$$

Deviatoric Stress Rate:

$$\dot{s} = 2G [\dot{\epsilon} - \dot{\epsilon}_p] = 2G [\dot{\epsilon} - \dot{\lambda} s]$$

Plastic Multiplier:

$$\dot{\lambda} = \frac{s : \dot{\epsilon}}{s : s}$$
RADIAL RETURN METHOD OF J₂-PLASTICITY II

Incremental Format:

\[ \Delta s = 2G [\Delta e - \Delta \lambda s] \]

Elastic Predictor-Plastic Corrector Split:

(a) Elastic Predictor: \( s_{trial} = s_n + 2G\Delta e \)
(b) Plastic Corrector: \( s_{n+1} = s_{trial} - 2G\Delta \lambda s_{trial} = [1 - 2G\Delta \lambda]s_{trial} \)

“Full” Consistency: \( F_{n+1} = \frac{1}{2} s_{n+1} : s_{n+1} - \frac{1}{3} \sigma_Y^2 = 0 \)

Quadratic equation for computing plastic multiplier \( \Delta \lambda_1 = ? \) and \( \Delta \lambda_2 = ? \)

\[ \frac{1}{2} [s_{trial} - 2G\Delta \lambda s_{trial}] : [s_{trial} - 2G\Delta \lambda s_{trial}] = \frac{1}{3} \sigma_Y^2 \]

Plastic Multiplier: \( \Delta \lambda_{min} = \frac{1}{2G} [1 - \sqrt{\frac{2}{3} \frac{\sigma_Y}{\sigma_{trial}:\sigma_{trial}}} ] \)

“Radial Return”: represents closest point projection of the trial stress state onto the yield surface. Final stress state is the scaled-back trial stress,

\[ s_{n+1} = \sqrt{\frac{2}{3} \frac{\sigma_Y}{s_{trial} : s_{trial}}} s_{trial} \]
GENERAL FORMAT OF PLASTIC RETURN METHOD

Incremental Format:
\[ \Delta \sigma = E : [\Delta \epsilon - \Delta \lambda m] \]

Elastic Predictor-Plastic Corrector Split:

(a) Elastic Predictor: \( \sigma_{\text{trial}} = \sigma_n + E : \Delta \epsilon \)
(b) Plastic Corrector: \( \sigma_{n+1} = \sigma_{\text{trial}} - \Delta \lambda E : m \)

“Full” Consistency: \( F_{n+1} = F(\sigma_n + E : \Delta \epsilon - \Delta \lambda E : m) = 0 \)

(i) Explicit Format: \( m = m_n \) (or evaluate \( m \) at \( m = m_c \) or \( m = m_{\text{trial}} \))

Use N-R for solving \( \Delta \lambda = ? \) for a given direction of plastic return e.g. \( m = m_n \).

(i) Implicit Format: \( m = m_{n+\alpha} \) where \( 0 < \alpha \leq 1 \) (\( m = m_{n+1} \) for BEM).

\[ F_{n+1} = F(\sigma_n + E : \Delta \epsilon - \Delta \lambda E : m_{n+\alpha}) = 0 \]

Use N-R for solving \( \Delta \lambda = ? \) in addition to unknown \( m = m_{n+\alpha} \).
ALGORITHMIC TANGENT STIFFNESS

Consistent Tangent vs Continuum Tangent:

Uniaxial Example:

\[ \dot{\sigma} = E_{\text{tan}} \dot{\epsilon} \quad \text{where} \quad E_{\text{tan}} = E_0 \left[ 1 - \frac{\sigma}{\sigma_0} \right] \quad \text{hence} \quad \sigma = \sigma_0 \left[ 1 - e^{-\frac{E_0}{\sigma_0} \dot{\epsilon}} \right] \]

Fully Implicit Euler Backward Integration:

\[ E_{\text{tan}} = E_{\text{tan}}^{n+1} \quad \text{in} \quad \Delta t = t_{n+1} - t_n, \]

\[ \sigma^{n+1} = \sigma^n + E_0 \left[ 1 - \frac{\sigma^{n+1}}{\sigma_0} \right] [\epsilon^{n+1} - \epsilon^n] \]

Algorithmic Tangent Stiffness: Relates \( d\epsilon^{n+1} \) to \( d\sigma^{n+1} \) at \( t_{n+1} \)

\[ d\sigma^{n+1} = E_{\text{tan}}^{alg} d\epsilon^{n+1} \quad \text{where} \quad E_{\text{tan}}^{alg} = \frac{E_0 \left[ 1 - \frac{\sigma^{n+1}}{\sigma_0} \right]}{1 + \frac{E_0}{\sigma_0} \Delta \epsilon} \]

Ratio of Tangent Stiffness Properties:

\[ \frac{E_{\text{tan}}^{alg}}{E_{\text{tan}}} = \frac{1}{1 + \frac{E_0}{\sigma_0} \Delta \epsilon} \sim 0.7 \]
ALGORITHMIC TANGENT STIFFNESS OF $J_2$-PLASTICITY

Incremental Form of Elastic-Plastic Split:

$$\Delta s = 2G[\Delta e - \Delta \lambda s]$$

Fully Implicit Euler Backward Integration: for $\Delta t = t_{n+1} - t_n$,

$$s_{n+1} = s_n + 2G[e_{n+1} - e_n] - \Delta \lambda 2G s_{n+1}$$

Relating $ds_{n+1}$ to $de_{n+1}$ at $t_{n+1}$, differentiation yields,

$$ds_{n+1} = 2G de_{n+1} - d\Delta \lambda 2G s_{n+1} - \Delta \lambda 2G ds_{n+1}$$

Algorithmic Tangent Stiffness Relationship:

$$ds_{n+1} = \frac{2G}{1 + \Delta \lambda 2G}[I - \frac{s_{n+1} \otimes s_{n+1}}{s_{n+1} : s_{n+1}}] : de_{n+1}$$

Ratio of Tangent Stiffness Properties: $\frac{G_{\text{alg}}}{G_{\text{cont}}} = \frac{1}{1 + \Delta \lambda 2G}$ when $||s_{n+1}|| \sim ||s_n||$.  

Class #5 Concrete Modeling, UNICAMP, Campinas, Brazil, August 20-28, 2007
CONCLUDING REMARKS

Main Lessons from Class # 5:

Nonlinear Solvers:

Forward Euler method introduces drift from true response path.
Newton-Raphson Iteration exhibits convergence difficulties when
\( K_{tan} \rightarrow 0 \) (ill-conditioning).

CPPM for Computational Plasticity:

Analytical Radial Return solution available for \( J_2 \)-plasticity and
Drucker-Prager. Generalization leads to explicit and implicit plastic
return strategies which are nowadays combined with the incremental
hardening and incremental stress residuals in a monolithic Newton
strategy.

Algorithmic vs Continuum Tangent:

For quadratic convergence tangent operator must be consistent with
integration of constitutive equations. The algorithmic tangent
compares to secant stiffness in increments which are truly finite.