# II. Buckingham Pi Theorem

The Buckingham Pi Theorem reduces a number of dimensional variables into a smaller number of dimensionless groups. The resulting dimensionless groups are <u>independent</u> but <u>not unique</u>. Additional dimensionless groups may be formed by taking products of the existing dimensionless groups.

The basic steps involved are listed below:

① Conjecture a physical relationship of the form

$$f(v_1, v_2, v_3, \ldots, v_m) = 0$$

where $v_i$ $1 \leq i \leq m$ are all of the dimensional parameters that have a bearing on the problem at hand.

② List the dimensions of each variable

$r =$ number of primary dimensions in the list

③ Form the matrix system

$$
\begin{matrix}
 & v_1\ v_2\ v_3\ \cdots\ v_m & \\
\begin{matrix} d_1 \\ d_2 \\ \vdots \\ d_r \end{matrix}
\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{m-1} \\ a_m \end{bmatrix}
= 0
\end{matrix}
$$

$i^{th}$ dimension in the problem (e.g. $M, L, \theta, T$)

Note: You must order the columns such that columns $v_1$ through $v_r$ contain at least one instance of each of the $r$ dimensions.

④ Perform Gaussian elimination. If a zero row appears, $r \leftarrow r-1$

⑤ Write expressions $a_1$ , $a_r$ in terms of

, $a_{r+2}, a_{r+3},$ , $a_m$ .

6. Write the equations from step ⑤ as

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} b^{I} \end{bmatrix} a_{r+1} + \begin{bmatrix} b^{II} \end{bmatrix} a_{r+2} + \quad \cdots \quad + \begin{bmatrix} b^{m-r} \end{bmatrix} a_m
$$

7. There are $m-r$ dimensionless groups defined by

$$
V_1^{b_1} \; V_2^{b_2} \; V_3^{b_3} \; \cdots \; V_{m-1}^{b_{m-1}} \; V_m^{b_m}
$$

Each $\vec{b}$ defines one dimensionless group.

## BUCKINGHAM PI EXAMPLE

Suppose we wish to investigate flow patterns in a lake, and we are interested in determining the nondimensional parameters associated with the problem.

Based on experience, we decide that the following dimensional variables are relevant:

$D_1$     length of lake  ($x_1$-direction)
$D_2$     width of lake  ($x_2$-direction)
$D_3$     depth of lake  ($x_3$-direction)
$U_1$     flow velocity in $x_1$ direction
$U_2$      "         "         "    $x_2$    "
$U_3$      "         "         "    $x_3$    "
$\rho$     density of lake water
$f$       coriolis parameter
$\nu$     lake water viscosity
$g$       gravitational constant
$t$       time
$\Delta p$    pressure

Following the steps outlined before, we have

① $\quad f\left(D_1, D_2, D_3, U_1, U_2, U_3, \rho, f, \nu, g, t, \Delta p\right) = 0 \qquad \boxed{m = 12}$

② $\quad [D_1] = [D_2] = [D_3] = L$

$\quad [U_1] = [U_2] = [U_3] = LT^{-1}$

$\quad [\rho] = ML^{-3}$

$\quad [f] = T^{-1}$

$\quad [\nu] = L^2 T^{-1}$

$\quad [g] = LT^{-2}$

$\quad [t] = T$

$\quad [\Delta p] = ML^{-1}T^{-2}$

3 dimensions in the list $(M, L, T)$, so $\boxed{r = 3}$

③

|  | $D_1$ | $U_1$ | $\rho$ | $f$ | $\nu$ | $g$ | $t$ | $\Delta p$ | $D_2$ | $D_3$ | $U_2$ | $U_3$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| L | 1 | 1 | -3 | 0 | 2 | 1 | 0 | -1 | 1 | 1 | 1 | 1 | | |
| T | 0 | -1 | 0 | -1 | -1 | -2 | 1 | -2 | 0 | 0 | -1 | -1 | | |
| M | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | |

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{bmatrix} = 0$$

④ Perform Gaussian elimination. Note that I have chosen $V_1$, $V_2$, and $V_3$ (r=3) so that this step is trivial:

|  | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ | $V_6$ | $V_7$ | $V_8$ | $V_9$ | $V_{10}$ | $V_{11}$ | $V_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| L | 1 | 1 | -3 | 0 | 2 | 1 | 0 | -1 | 1 | 1 | 1 | 1 |
| T | 0 | 1 | 0 | 1 | 1 | 2 | -1 | 2 | 0 | 0 | 1 | 1 |
| M | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

there are no zero rows, so r is indeed 3.

⑤ from row 3 we have: $a_3 + a_8 = 0$ ⟹ $\boxed{a_3 = -a_8}$

from row 2 we have:

$a_2 + a_4 + a_5 + 2a_6 - a_7 + 2a_8 + a_{11} + a_{12} = 0$

⟹ $\boxed{a_2 = -a_4 - a_5 - 2a_6 + a_7 - 2a_8 - a_{11} - a_{12}}$

from row 1 we have:

$a_1 + a_2 - 3a_3 + 2a_5 + a_6 - a_8 + a_9 + a_{10} + a_{11} + a_{12} = 0$

so $a_1 = -\left(-a_4 - a_5 - 2a_6 + a_7 - 2a_8 - a_{11} - a_{12}\right) + 3\left(-a_8\right) - 2a_5 - a_6 + a_8 - a_9 - a_{10} - a_{11} - a_{12}$

⟹ $\boxed{a_1 = a_4 - a_5 + a_6 - a_7 - a_9 - a_{10}}$

since r = 3, we have written $a_1$, $a_2$, and $a_3$ in terms of $a_4$, $a_5$, $a_6$, ..., $a_{10}$, $a_{11}$, $a_{12}$.

(6.)

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{bmatrix}
=
\overset{b^{I}}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} a_4 +
\overset{b^{II}}{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} a_5 +
\overset{b^{III}}{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} a_6 +
\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} a_7 +
\begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} a_8 +
\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} a_9 +
\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} a_{10} +
\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} a_{11} +
\overset{b^{IX}}{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} a_{12}
$$

(7.)   There are   $m - r = 9$   nondimensional groups

For example, the first one is   $V_1' \, V_2^{-1} \, V_4' = \dfrac{V_1 V_4}{V_2} = \dfrac{D_1 f}{U_1}$

The complete set of dimensionless parameters is given below:
(Note that I have inverted some of them)

$\dfrac{U_1}{f D_1}$     "Rossby Number"          ratio of inertial forces to Coriolis "forces"

$\dfrac{U_1 D_1}{\nu}$     "Reynolds Number"          ratio of inertial forces to viscous diffusion forces

$\dfrac{U_1^2}{g D_1}$     "Froude Number"          ratio of inertial forces to gravity forces

$\dfrac{D_1}{U_1 t}$     "Strouhal Number"          ratio of inertial forces to forces from local accelerations
                                                      (unsteadiness)

$\dfrac{\Delta p}{\rho U_1^2}$     "Euler Number"          ratio of inertial forces to pressure forces.

$\left.\begin{array}{c} \dfrac{D_2}{D_1} \\[1em] \dfrac{D_3}{D_1} \end{array}\right\}$   Aspect Ratios

$\left.\begin{array}{c} \dfrac{U_2}{U_1} \\[1em] \dfrac{U_3}{U_1} \end{array}\right\}$   Velocity Ratios