MODELING OF CONCRETE MATERIALS AND STRUCTURES

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Class Meeting #4: Failure Analysis at Constitutive Level

Continuous vs Discontinuous Failure: $Continuum \Rightarrow Discontinuum$

Loss of Material Stability/Uniqueness: $d^2W = 0 \ vs \ \det \boldsymbol{\mathcal{E}} = 0$

Loss of Ellipticity/Hyperbolicity: $Localization \ Analysis$

Geometric Localization Criterion: $Elliptic \ Localization \ Envelope$

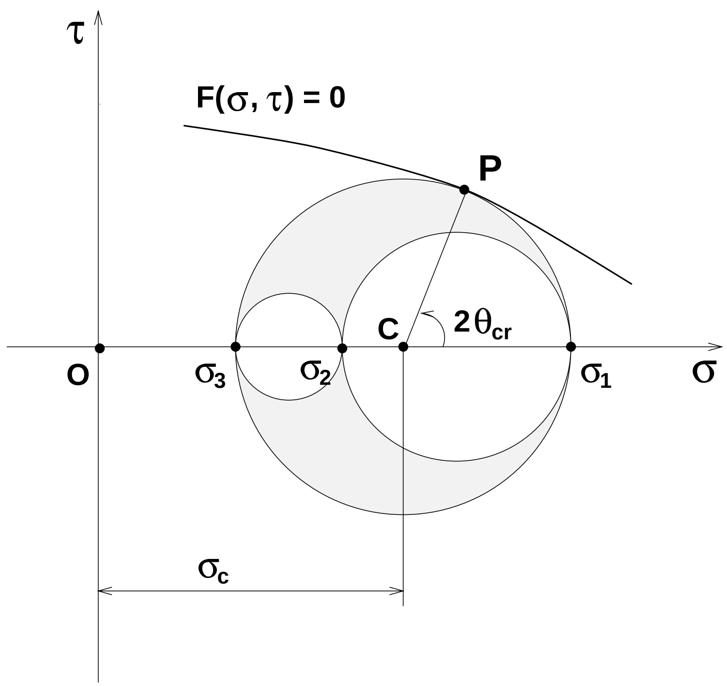
MOHR FAILURE ENVELOPE

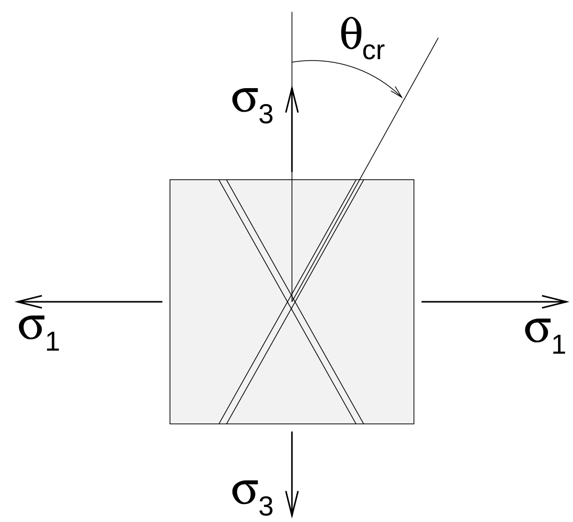
Geometric Concept: $F(\sigma, \tau) = f(\boldsymbol{\sigma}) - r_y = 0$

O. Mohr [1900]: Critical Mohr circle contacts failure envelope

$$f(\boldsymbol{\sigma}) = \frac{1}{2}[\sigma_1 - \sigma_3]$$

Is there a universal ‘strength criterion’ for brittle-ductile failure





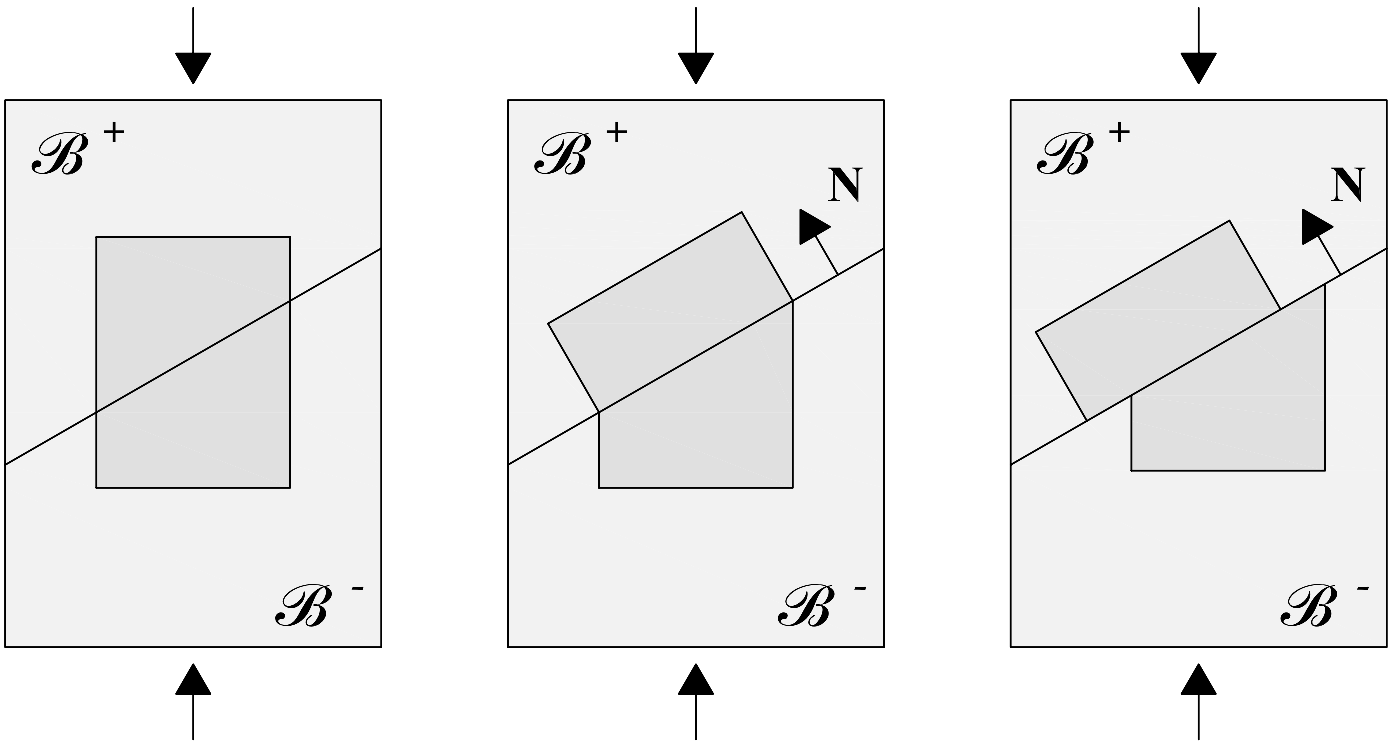
DEGRADATION OF KINEMATIC CONTINUITY

Transition of Continuum into Discontinuum

1. Diffuse Failure : continuous velocities and velocity gradients

2. Localized Failure : formation of weak discontinuities

3. Discrete Failure : formation of strong discontinuities



BOND AT BIMATERIAL INTERFACE

Perfect Contact:

$$[|\mathbf{u}_N|] = \mathbf{u}_N^b - \mathbf{u}_N^m = \mathbf{0} \quad \text{and} \quad [|\mathbf{t}_N|] = \mathbf{t}_N^b - \mathbf{t}_N^m = \mathbf{0}$$

Weak Discontinuities: all strain components exhibit jumps across interface except for $\epsilon_{TT}^b = \epsilon_{TT}^m$ restraint.

Note: Jump of tangential normal stress, $\sigma_{TT}^b \neq \sigma_{TT}^m$.

Imperfect Contact:

$$[|\mathbf{u}_N|] = \mathbf{u}_N^b - \mathbf{u}_N^m \neq \mathbf{0} \quad \text{whereas} \quad [|\mathbf{t}_N|] = \mathbf{t}_N^b - \mathbf{t}_N^m = \mathbf{0}$$

Strong Discontinuities: all displacement components exhibit jumps across interface.

Note: FE Displacement method enforces traction continuity in 'weak' sense only, hence $[|\mathbf{t}_N|] \neq 0$.

MATERIAL STABILITY

Second Order Work Density Functional:

$$d^2W = \frac{1}{2}\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}} = \frac{1}{2}\dot{\boldsymbol{\epsilon}} : \boldsymbol{\mathcal{E}}_{tan} : \dot{\boldsymbol{\epsilon}} > 0, \forall \dot{\boldsymbol{\epsilon}} \neq 0$$

For non-associated plasticity:

$$d^2W = \frac{1}{2}\dot{\boldsymbol{\epsilon}} : \boldsymbol{\mathcal{E}} : \dot{\boldsymbol{\epsilon}} - \frac{1}{4h_p}\dot{\boldsymbol{\epsilon}} : [\bar{\boldsymbol{m}} \otimes \bar{\boldsymbol{n}} + \bar{\boldsymbol{n}} \otimes \bar{\boldsymbol{m}}] : \dot{\boldsymbol{\epsilon}}$$

Note: The energy functional uses only the symmetric tangent operator.

Bromwich Eigenvalue Bounds:

$$\lambda_{min}(\boldsymbol{\mathcal{E}}_{ep}^{sym}) \leq \mathbb{R}(\lambda_{min}(\boldsymbol{\mathcal{E}}_{ep})).. \leq \lambda_{max}(\boldsymbol{\mathcal{E}}^{sym})$$

Material instability coincides with loss of positive definiteness of the symmetric operator: $\lambda_{min}(\boldsymbol{\mathcal{E}}_{ep}^{sym}) = 0$

Critical Hardening Modulus, [Maier & Hueckl, 1979]:

$$H_p^{stabil} = \frac{1}{2}\left[\sqrt{(\boldsymbol{n} : \boldsymbol{\mathcal{E}} : \boldsymbol{n})(\boldsymbol{m} : \boldsymbol{\mathcal{E}} : \boldsymbol{m})} - \boldsymbol{n} : \boldsymbol{\mathcal{E}} : \boldsymbol{m}\right]$$

MATERIAL UNIQUENESS

Loss of Material Uniqueness: $\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{E}}_{ep} : \dot{\boldsymbol{\epsilon}} = \mathbf{0}$

Indicates stationary stress state at limit point. Loss of uniqueness is synonymous with the formation of a singular tangent operator.

$$\det \boldsymbol{\mathcal{E}}_{ep} \stackrel{!}{=} 0 \qquad \rightarrow \qquad \lambda_{min}(\boldsymbol{\mathcal{E}}_{ep}) = 0$$

The plastic operator is a rank-one update of the positive elasticity tensor,

$$\boldsymbol{\mathcal{E}}_{ep} = \boldsymbol{\mathcal{E}} - \frac{1}{h_p} \bar{\boldsymbol{m}} \otimes \bar{\boldsymbol{n}}$$

Pre–conditioning

$$\boldsymbol{\mathcal{E}}^{-1} : \boldsymbol{\mathcal{E}}^{ep} = \boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{E}}^{-1} : \frac{\bar{\boldsymbol{m}} \otimes \bar{\boldsymbol{n}}}{h_p}$$

Critical eigenvalue λ_{min} measures uniqueness by scalar damage variable $d_{\mathcal{E}}$,

$$\lambda_{min}\left(\boldsymbol{\mathcal{E}}^{-1} : \boldsymbol{\mathcal{E}}^{ep}\right) = 1 - d_{\mathcal{E}} \qquad \text{with} \qquad d_{\mathcal{E}} := \frac{\boldsymbol{n} : \boldsymbol{\mathcal{E}} : \boldsymbol{m}}{H_p + \boldsymbol{n} : \boldsymbol{\mathcal{E}} : \boldsymbol{m}}$$

Note: $H_p^{limit} = 0$ corresponds to $1 - d_{\mathcal{E}} \stackrel{!}{=} 0$ or to $d_{\mathcal{E}} = 1$.

LOCALIZATION ANALYSIS

Kinematic Compatibility across Discontinuity Surface:

$$[|\nabla \dot{\boldsymbol{u}}|] = \boldsymbol{M} \otimes \boldsymbol{N} \qquad \rightarrow \qquad [|\dot{\boldsymbol{\epsilon}}|] = \frac{1}{2} \left[\boldsymbol{M} \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \boldsymbol{M} \right]$$

Traction Equilibrium: Cauchy's Theorem

$$[|\dot{\boldsymbol{t}}|] = \dot{\boldsymbol{t}}^{+} - \dot{\boldsymbol{t}}^{-} = \boldsymbol{0}$$

$$[|\dot{\boldsymbol{t}}|] = \boldsymbol{N} \cdot [|\dot{\boldsymbol{\sigma}}|] = \boldsymbol{N} \cdot [|\boldsymbol{\mathcal{E}}_{tan} : \dot{\boldsymbol{\epsilon}}|] = \boldsymbol{0}$$

Assuming $[|\boldsymbol{\mathcal{E}}_{tan}|] = \boldsymbol{\mathcal{E}}^{+}_{tan} - \boldsymbol{\mathcal{E}}^{-}_{tan} = \boldsymbol{0}$

Continuous Material Bifurcation:

$$\boldsymbol{Q}_{tan} \cdot \boldsymbol{M} = \boldsymbol{0} \qquad \text{with} \qquad \boldsymbol{Q}_{tan} = \boldsymbol{N} \cdot \boldsymbol{\mathcal{E}}_{tan} \cdot \boldsymbol{N}$$

$\boldsymbol{Q}_{tan}$ is the tangential localization tensor with Localization Criterion:

$$\det \boldsymbol{Q}_{tan} \stackrel{!}{=} 0 \qquad \rightarrow \qquad \lambda_{min}(\boldsymbol{Q}_{tan}) = 0$$

ELASTOPLASTIC LOCALIZATION CONDITION

Elastoplastic Bifurcation Condition:

$$\det(\boldsymbol{Q}_{ep}) = \det(\boldsymbol{N} \cdot [\boldsymbol{E}_o - \frac{1}{h_p}\boldsymbol{E}_o : \boldsymbol{m} \otimes \boldsymbol{n} : \boldsymbol{E}_o] \cdot \boldsymbol{N}) = 0$$

Rank–one Update Format of Elastoplastic Localization Tensor:

$$\boldsymbol{Q}_{ep} = \boldsymbol{Q}_0 - \frac{1}{h_p}\boldsymbol{e}_m \otimes \boldsymbol{e}_n$$

where

$$\begin{aligned} \boldsymbol{e}_m &= \boldsymbol{N} \cdot \boldsymbol{\mathcal{E}} : \boldsymbol{m} \\ \boldsymbol{e}_n &= \boldsymbol{n} : \boldsymbol{\mathcal{E}} \cdot \boldsymbol{N} \end{aligned}$$

Discontinuous Failure Mode: $[|\dot{\boldsymbol{\epsilon}}|] = \frac{1}{2}\ [\ \boldsymbol{M} \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \boldsymbol{M}\ ]$

Failure orientation depends on: $\boldsymbol{m} = \frac{\partial Q}{\partial \boldsymbol{\sigma}}$ and $\boldsymbol{n} = \frac{\partial F}{\partial \boldsymbol{\sigma}}$

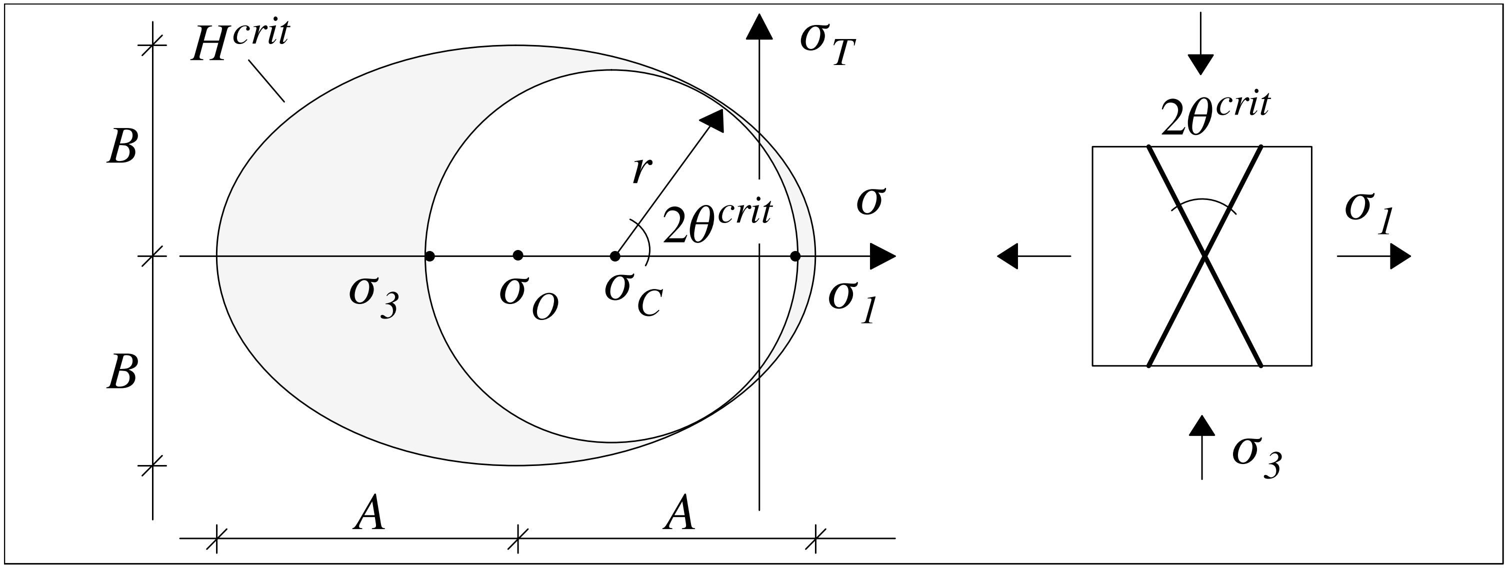
LOCALIZATION ELLIPSE

Scalar Form of Localization Condition:

$$H_p^{loc} + \boldsymbol{n} : \boldsymbol{\mathcal{E}} : \boldsymbol{m} = \boldsymbol{e}_n \cdot \boldsymbol{Q}^{-1} \cdot \boldsymbol{e}_m$$

Geometric Envelope Condition:

$$\frac{(\sigma - \sigma_0)^2}{A^2} + \frac{\tau^2}{B^2} = 1$$



LOCALIZED FAILURE MODE

Parabolic Drucker-Prager:

$F = J_2 + \alpha_F\, I_1 - \beta_F$ and $Q = J_2 + \alpha_Q\, I_1 - \beta_Q$

Character of Jump: $[|\dot{\boldsymbol{\epsilon}}|] = \frac{1}{2}\ [\ \boldsymbol{M} \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \boldsymbol{M}\ ]$

Half Axes of Localization Ellipse:

$A^2 = \frac{2(1-\nu)}{1-2\nu} B^2$ and $B^2 = \frac{1}{4G} H_p^{loc} + J_2 + \frac{1-\nu}{8(1-2\nu)}(\alpha_F + \alpha_Q)^2 + \frac{1+2\nu}{1-2\nu}\alpha_F\alpha_Q$

Critical Normal Vector of Failure Plane $\boldsymbol{N}$ w/r to major principal $\boldsymbol{e}_1$-axis:

$$tan^2\theta^{cr} = \frac{r - [(1-2\nu)(\sigma_c - \frac{1}{3}I_1) + \frac{1}{2}(1-\nu)(\alpha_F + \alpha_Q)}{r + [(1-2\nu)(\sigma_c - \frac{1}{3}I_1) + \frac{1}{2}(1-\nu)(\alpha_F + \alpha_Q)}$$

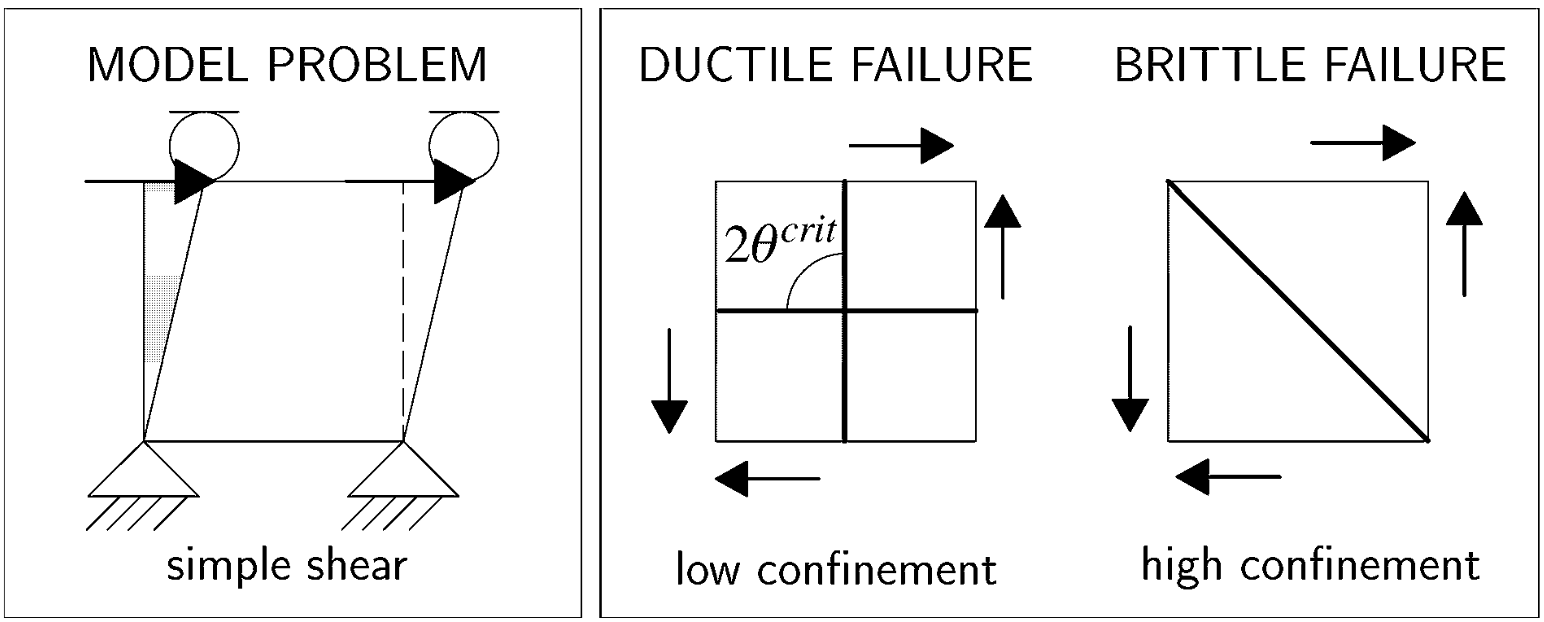
Critical Hardening Modulus:

$$H_p^{cr} = 4G\{r^2 + (1-2\nu)[\sigma_c - \frac{1}{3}I_1 + \frac{(1-\nu)(\alpha_F + \alpha_Q)}{2(1-2\nu)}]^2 - J_2 - \frac{(1-\nu)(\alpha_F + \alpha_Q)^2}{8(1-2\nu)} - \frac{(1+2\nu}{(1-2\nu}$$

MODEL PROBLEM OF SIMPLE SHEAR: $\dot{\gamma}_{12} > 0$

Non-Associated Parabolic Drucker-Prager Model: $\alpha_Q = 0$

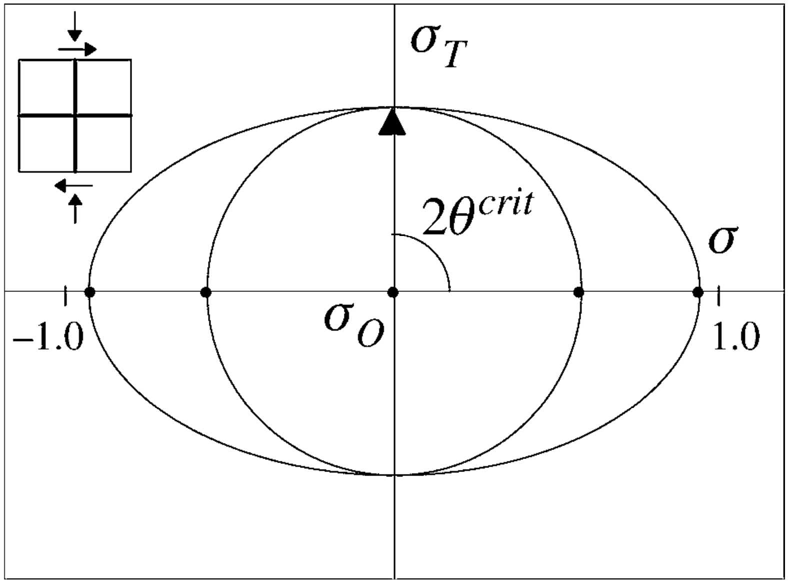
Failure Mode (angle θ^{crit}) and Contrast Strength Ratios $f_c' : f_t'$



MODEL PROBLEM OF SIMPLE SHEAR: $\dot{\gamma}_{12} > 0$

Geometric Localization Analysis: $f'_c : f'_t = 1 : 1$, von Mises

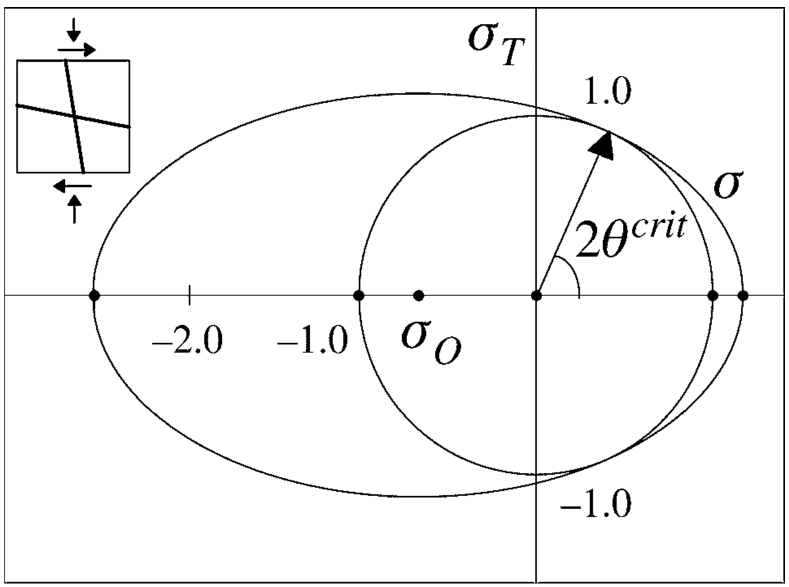
Pressure-Insensitive Failure: $\theta^{crit} = 45^0$ and $\theta^{crit} = 135^0$ shear failure mode II.



MODEL PROBLEM OF SIMPLE SHEAR: $\dot{\gamma}_{12} > 0$

Geometric Localization Analysis: $f'_c : f'_t = 3 : 1$

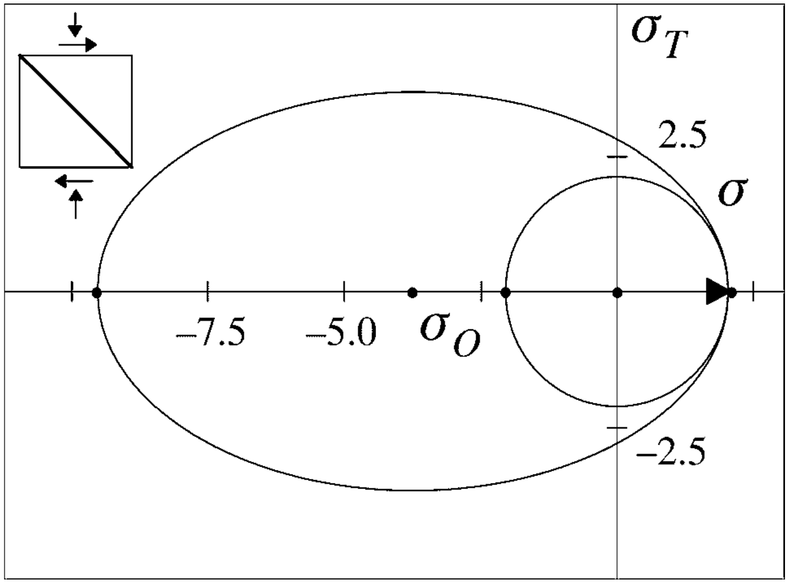
Pressure-Sensitive Failure: $\theta = 33.211^0$ and $\theta = 146.79^0$ indicate mixed shear-compression failure.



MODEL PROBLEM OF SIMPLE SHEAR: $\dot{\gamma}_{12} > 0$

Geometric Localization Analysis: $f'_c : f'_t = 12 : 1$

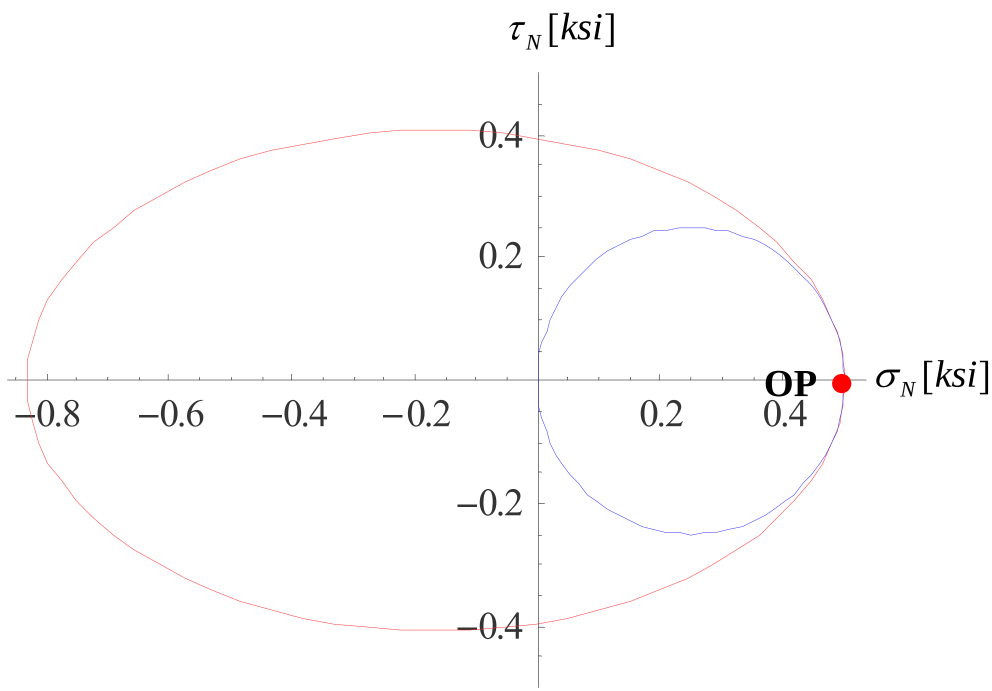
Highly Pressure-Sensitive Failure: $\theta^{crit} = 0^0$ indicates brittle failure mode I.

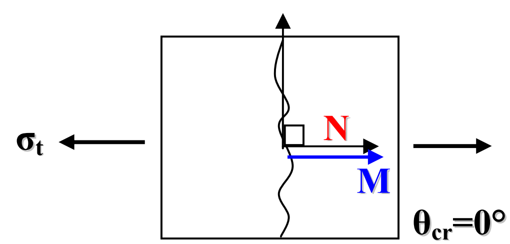


MODE I: SPLITTING TENSION

Critical Localization Condition: Mode I when $\boldsymbol{N}||\boldsymbol{M}$ and $\theta^{cr} = 0$

Associated Flow: $\alpha_F = \alpha_Q = 0.25$
Non-Associated Flow: $\alpha_F = 1.167, \alpha_Q = -0.667$

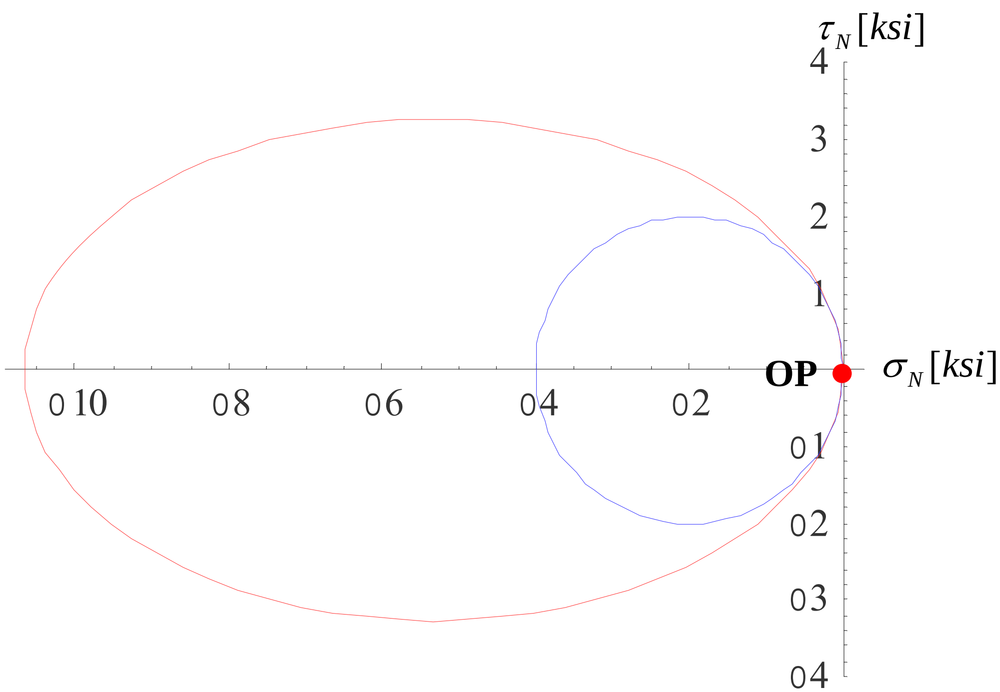


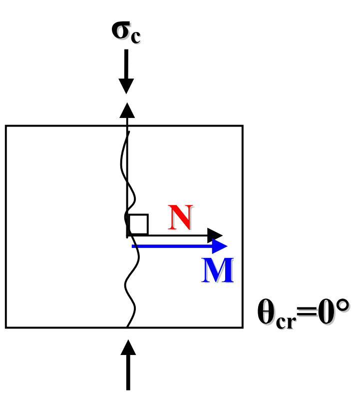


MODE I: SPLITTING COMPRESSION

Critical Localization Condition: Mode I when $\boldsymbol{N} || \boldsymbol{M}$ and $\theta^{cr} = 0$

Associated Flow: $\alpha_F = \alpha_Q = 3.0$
Non-Associated Flow: $\alpha_F = 1.167, \alpha_Q = 4.833$

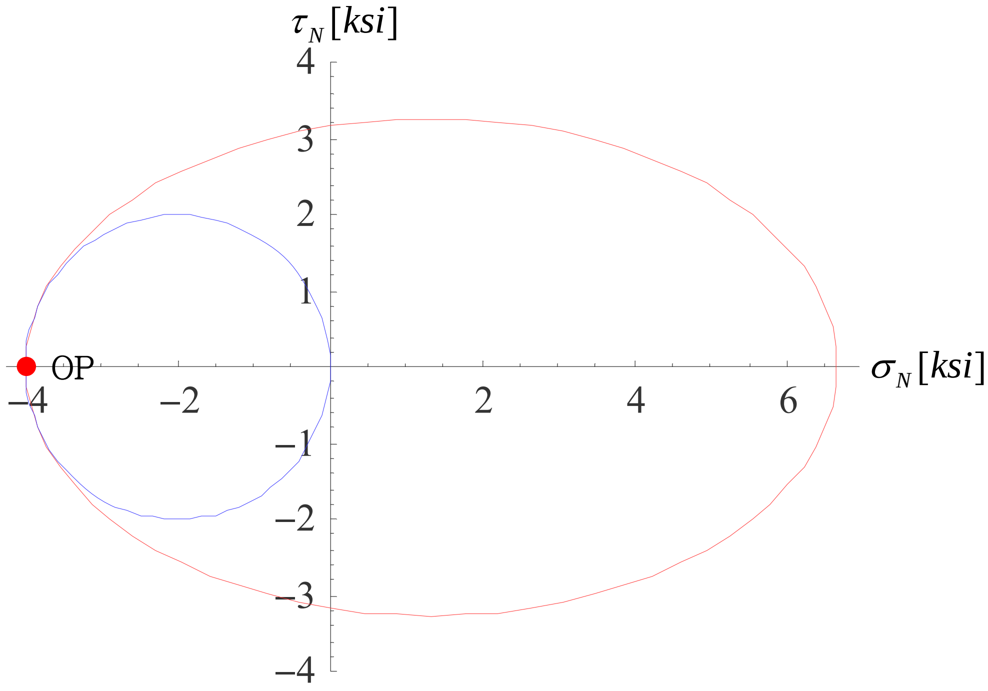




COMPACTION BANDING

Critical Localization Condition: Compaction Band

Associated Flow for Compaction Band : $\alpha_F = \alpha_Q = -2.0$



CONCLUDING REMARKS

Main Lessons from Class # 4:

Diffuse Failure:

Loss of Stability and Loss of Uniqueness

Localized Failure:

Loss of Ellipticity and Hyperbolicity

Volumetric-Deviatoric Coupling:

Simple Shear Test Exhibits Confinement Effects

Compression Failure of Brittle Materials:

Splitting Compression Depends on Confinement