# CVEN 5161

# Advanced Mechanics of Materials I

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Revised Version of Class Notes

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# Chapter 1

# Preliminaries

The mathematical tools behind stress and strain are housed in Linear Algebra and Vector and Tensor Analysis in particular. For this reason let us revisit established concepts which hopefully provide additional insight into matrix analysis beyond the mere mechanics of elementary matrix manipulation.

## 1.1 Vector and Tensor Analysis

### (a) Cartesian Description of Vectors:

Forces, displacements, velocities and accelerations are objects $\boldsymbol{F}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in \Re^3$ which may be represented in terms of a set of base vectors and their components. In the following we resort to cartesian coordinates in which the base vectors $\boldsymbol{e}_i$ form an orthonormal set which satisfies

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \quad \forall \quad i,j = 1,2,3 \quad \text{where} \quad [\delta_{ij}] = \boldsymbol{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1}$$

where the Kronecker symbol $[\delta_{ij}] = \boldsymbol{1}$ designates the unit tensor. Consequently, the force vector may be expanded in terms of its components and base vectors,

$$\boldsymbol{F} = F_1\boldsymbol{e}_1 + F_2\boldsymbol{e}_2 + F_3\boldsymbol{e}_3 = \sum_i^3 F_i\boldsymbol{e}_i \rightarrow F_i\boldsymbol{e}_i \tag{1.2}$$

The last expression is a short hand index notation in which repeated indices infer summation over 1,2,3.

In matrix notation, the inner product may be written in the form,

$$\boldsymbol{F} = [F_1, F_2, F_3]^t \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{bmatrix} \rightarrow [F_i] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{1.3}$$

where we normally omit the cartesian base vectors and simply represent a vector in terms of its components (coordinates).

**(b) Scalar Product (Dot Product) of Two Vectors**: $(\boldsymbol{F} \cdot \boldsymbol{u}) \in \Re^3$

The dot operation generates the scalar:

$$W = (\boldsymbol{F} \cdot \boldsymbol{u}) = (F_i \boldsymbol{e}_i) \cdot (u_j \boldsymbol{e}_j) = F_i u_j \delta_{ij} = F_i u_i \tag{1.4}$$

Using matrix notation, the inner product reads $\boldsymbol{F}^t \boldsymbol{u} = F_i u_i$, where $\boldsymbol{F}^t$ stands for a row vector and $\boldsymbol{u}$ for a column vector. The mechanical interpretation of the scalar product is a work or energy measure with the unit [1 J = 1Nm]. The magnitude of the dot product is evaluated according to,

$$W = |\boldsymbol{F}||\boldsymbol{u}| \cos \theta \tag{1.5}$$

where the absolute values $|\boldsymbol{F}| = (\boldsymbol{F} \cdot \boldsymbol{F})^{\frac{1}{2}}$ and $|\boldsymbol{u}| = (\boldsymbol{u} \cdot \boldsymbol{u})^{\frac{1}{2}}$ are the Euclidean lengths of the two vectors. The angle between two vectors is defined as

$$\cos \theta = \frac{(\boldsymbol{F} \cdot \boldsymbol{u})}{|\boldsymbol{F}||\boldsymbol{u}|} \tag{1.6}$$

The two vectors are orthogonal when $\theta = \pm \frac{\pi}{2}$,

$$(\boldsymbol{F} \cdot \boldsymbol{u}) = 0 \quad \text{if} \quad \boldsymbol{F} \neq \boldsymbol{0} \quad \text{and} \quad \boldsymbol{u} \neq \boldsymbol{0} \tag{1.7}$$

In other terms there is no work done when the force vector is orthogonal to the displacement vector.

Note: Actually, the mechanical work is the line integral of the scalar product between the force vector times the displacement rate, $W = \int \boldsymbol{F} \cdot d\boldsymbol{u}$. Hence the previous expressions were based on the assumption that the change of work along the integration path is constant.

**(c) Vector Product (Cross Product) of Two Vectors**: $(\boldsymbol{x} \times \boldsymbol{y}) \in \Re^3$:

The cross product generates the vector $\boldsymbol{z}$ which is orthogonal to the plane spanned by the two vectors $\boldsymbol{x}, \boldsymbol{y}$ according to the right hand rule:

$$\boldsymbol{z} = \boldsymbol{x} \times \boldsymbol{y} = x_i y_j (\boldsymbol{e}_i \times \boldsymbol{e}_j) \quad \text{where} \quad |\boldsymbol{z}| = |\boldsymbol{x}||\boldsymbol{y}| \sin \theta \tag{1.8}$$

In 3-d the determinant scheme is normally used for the numerical evaluation of the cross product. Its components may be written in index notation with respect to cartesian coordinates

$$z_p = \epsilon_{pqr} x_q y_r \tag{1.9}$$

The permutation symbol $\epsilon_{pqr} = 0, 1, -1$ is zero when any two indices are equal, it is one when $p, q, r$ are even permutations of $1, 2, 3$, and it is minus one when they are odd.

The geometric interpretation of the cross product is the area spanned by the two vectors, $|\boldsymbol{z}| = A$. In other terms the determinant is a measure of the area subtended by the two vectors $\boldsymbol{x} \times \boldsymbol{y}$, where

$$\boldsymbol{z} = \det \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \tag{1.10}$$

The scalar triple product $V = \boldsymbol{z} \cdot (\boldsymbol{x} \times \boldsymbol{y})$ measures the volume spanned by the three vectors:

$$V = |\boldsymbol{z}| cos\psi |\boldsymbol{x}||\boldsymbol{y}| sin\theta = \det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \tag{1.11}$$

**(d) Tensor Product (Dyadic Product) of Two Vectors**: $\boldsymbol{u} \otimes \boldsymbol{v} \in \Re^3$:

The tensor product of two vectors generates a second order tensor (i.e. a matrix of rank-one) $\boldsymbol{A}^{(1)}$:

$$\boldsymbol{A}^{(1)} = \boldsymbol{u} \otimes \boldsymbol{v} = u_i \boldsymbol{e}_i \otimes v_j \boldsymbol{e}_j = u_i v_j \boldsymbol{e}_i \otimes \boldsymbol{e}_j = [u_i v_j] \tag{1.12}$$

In matrix notation the tensor product writes as $\boldsymbol{u} \otimes \boldsymbol{v} = [u_i v_j] = \boldsymbol{u}\boldsymbol{v}^t$. Expanding, we have

$$\boldsymbol{u} \otimes \boldsymbol{v} = [u_i v_j] = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix} \tag{1.13}$$

The linear combination of three dyads is called a 'dyadic' which may be used to generate a second order tensor of full rank three, $\boldsymbol{A}^{(3)}$:

$$\boldsymbol{A}^{(3)} = \sum_{k=1}^{3} a_k \boldsymbol{u}_k \otimes \boldsymbol{v}_k \tag{1.14}$$

Spectral representation of a second order tensor $\boldsymbol{A}^{(3)}$ expresses the tensor in terms of eigenvalues $\lambda_i$ and eigenvectors. They form an orthogonal frame which may be normalized by $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \boldsymbol{1}$ such that,

$$\boldsymbol{A}^{(3)} = \sum_{i=1}^{3} \lambda_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i \tag{1.15}$$

Note: the tensor product increases the order of of the resulting tensor by a factor two. Hence the tensor product of two vectors (two tensors of order one) generates a second order tensor, and the tensor product of two second order tensors generates a fourth order tensor, etc.

**(e) Coordinate Transformations:**

Transformation of the components of a vector from a proper orthonormal coordinate system $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ into another proper orthonormal system $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3$ involves the operator $\boldsymbol{Q}$ of direction cosines. In 3-d we have,

$$\boldsymbol{Q} = \cos(\boldsymbol{E}_i \cdot \boldsymbol{e}_j) \tag{1.16}$$

where the matrix of direction cosines is comprised of,

| | $\boldsymbol{e}_1$ | $\boldsymbol{e}_2$ | $\boldsymbol{e}_3$ |
|---|---|---|---|
| $\boldsymbol{E}_1$ | $\cos(E_1 e_1)$ | $\cos(E_1 e_2)$ | $\cos(E_1 e_3)$ |
| $\boldsymbol{E}_2$ | $\cos(E_2 e_1)$ | $\cos(E_2 e_2)$ | $\cos(E_2 e_3)$ |
| $\boldsymbol{E}_3$ | $\cos(E_3 e_1)$ | $\cos(E_3 e_2)$ | $\cos(E_3 e_3)$ |

(1.17)

The $\boldsymbol{Q}$-operator forms a proper orthonormal transformation, $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^t$ i.e. it satisfies the conditions,

$$\boldsymbol{Q}^t \cdot \boldsymbol{Q} = \mathbf{1} \quad \text{and} \quad \det \boldsymbol{Q} = 1 \tag{1.18}$$

In the case of 2-d, this transformation results in a plane rotation around the $x_3$-axis, i.e.

$$\boldsymbol{Q} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.19}$$

where $\theta \geq 0$ for counterclockwise rotations according the right hand rule. Consequently, the transformation of the components of the vector $\boldsymbol{F}$ from one coordinate system into another,

$$\tilde{\boldsymbol{F}} = \boldsymbol{Q} \cdot \boldsymbol{F} \quad \text{and inversely} \quad \boldsymbol{F} = \boldsymbol{Q}^t \cdot \tilde{\boldsymbol{F}} \tag{1.20}$$

The $(3 \times 3)$ dyadic forms an array $\boldsymbol{A}$ of nine scalars which follows the transformation rule of second order tensors,

$$\tilde{\mathbf{A}} = \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^t \quad \text{and inversely} \quad \mathbf{A} = \mathbf{Q}^t \cdot \tilde{\mathbf{A}} \cdot \mathbf{Q} \tag{1.21}$$

This distinguishes the array $\boldsymbol{A}$ to form a second order tensor.

**(f) The Stress Tensor:**

Considering the second order stress tensor as an example when $\boldsymbol{A} = \boldsymbol{\sigma}$ then the linear transformation results in the Cauchy theorem, that mapping of the stress tensor onto the plane with the normal $\boldsymbol{n}$ results in the traction vector $\boldsymbol{t}$:

$$\boldsymbol{\sigma}^t \cdot \boldsymbol{n} = \boldsymbol{t} \quad \text{or} \quad \sigma_{ji} n_j = t_i \tag{1.22}$$

This transformation may be viewed as a projection of the stress tensor onto the plane with the normal vector $\boldsymbol{n}^t = [n_1, n_2, n_3]$, where

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{1.23}$$

with the understanding that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$ is symmetric in the case of Boltzmann continua. The Cauchy theorem states that the state of stress is defined uniquely in terms of the traction vectors on three orthogonal planes $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3$ which form the nine entries in the stress tensor. Given the stress tensor, the traction vector is uniquely defined on any arbitrary plane with the normal $\boldsymbol{n}$, the components of which are,

$$\boldsymbol{t} = \boldsymbol{\sigma}^t \cdot \boldsymbol{n} = \begin{bmatrix} \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \\ \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \\ \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 \end{bmatrix} \tag{1.24}$$

The traction vector $\boldsymbol{t}$ may be decomposed into normal and tangential components on the plane $\boldsymbol{n}$,

$$\sigma_n = \boldsymbol{n} \cdot \boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma}^t \cdot \boldsymbol{n} \tag{1.25}$$

and

$$(\tau_n)^2 = |\boldsymbol{t}|^2 - \sigma_n^2 = \boldsymbol{t} \cdot \boldsymbol{t} - (\boldsymbol{n} \cdot \boldsymbol{t})^2 \tag{1.26}$$

which are the normal stress and the tangential stress components on that plane.

**(g) Eigenvalues and Eigenvectors:**

There exist a nonzero vector $\boldsymbol{n}_p$ such that the linear transformation $\boldsymbol{\sigma}^t \cdot \boldsymbol{n}_p$ is a multiple of $\boldsymbol{n}_p$. In this case $\boldsymbol{n} \| \boldsymbol{t}$ we talk about principal direction $\boldsymbol{n} = \boldsymbol{n}_p$ in which the tangential shear components vanish,

$$\boldsymbol{\sigma}^t \cdot \boldsymbol{n}_p = \lambda \boldsymbol{n}_p \tag{1.27}$$

The eigenvalue problem is equivalent to stating that

$$[\boldsymbol{\sigma}^t - \lambda \mathbf{1}] \cdot \boldsymbol{n}_p = \mathbf{0} \tag{1.28}$$

For a non-trivial solution $\boldsymbol{n}_p \neq \mathbf{0}$ the matrix $(\boldsymbol{\sigma}^t - \lambda \mathbf{1})$ must be singular. Consequently, $\det(\boldsymbol{\sigma}^t - \lambda \mathbf{1}) = 0$ generates the characteristic polynomial

$$p(\lambda) = \det(\boldsymbol{\sigma}^t - \lambda \mathbf{1}) = \lambda^3 - I_1 \lambda^2 - I_2 \lambda - I_3 = 0 \tag{1.29}$$

where the three principal invariants are,

$$I_1 = \mathrm{tr}\boldsymbol{\sigma} = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 \tag{1.30}$$

$$I_2 = \frac{1}{2}[tr\boldsymbol{\sigma}^2 - (\mathrm{tr}\boldsymbol{\sigma})^2] = \frac{1}{2}[\sigma_{ij}\sigma_{ij} - \sigma_{ii}^2] = -[\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1] \tag{1.31}$$

$$I_3 = \det \boldsymbol{\sigma} = \sigma_1\sigma_2\sigma_3 \tag{1.32}$$

According to the fundamental theorem of algebra, a polynomial of degree 3 has exactly 3 roots, thus each matrix $\boldsymbol{\sigma} \in \Re^3$ has 3 eigenvalues $\lambda_1 = \sigma_1, \lambda_2 = \sigma_2, \lambda_3 = \sigma_3$. If a polynomial with real coefficients has some non-real complex zeroes, they must occur in conjugate pairs.

Note: all three eigenvalues are real when the stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$ is symmetric.

Further, $\det(\boldsymbol{\sigma}^t - \lambda \mathbf{1}) = (-1)^3 \det(\lambda \mathbf{1} - \boldsymbol{\sigma}^t)$, thus the roots of both characteristic equations are the same.

**(i) Similarity Equivalence:**

Similarity transformations, i.e. triple products of the form

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{S}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{S} \tag{1.33}$$

preserve the spectral properties of $\boldsymbol{\sigma}$. In other terms, if $\tilde{\boldsymbol{\sigma}} \sim \boldsymbol{\sigma}$, then the characteristic polynomial of $\tilde{\boldsymbol{\sigma}}$ is the same as that of $\boldsymbol{\sigma}$.

$$p(\lambda) = \det(\tilde{\boldsymbol{\sigma}} - \lambda \mathbf{1}) = \det(\boldsymbol{S}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{S} - \lambda \boldsymbol{S}^{-1} \cdot \boldsymbol{S}) = \det(\boldsymbol{\sigma} - \lambda \mathbf{1}) \tag{1.34}$$

**(j) Orthonormal or Unitary Equivalence:**

A matrix $\boldsymbol{U} \in \Re^3$ is unitary (real orthogonal) if

$$\boldsymbol{U}^t \cdot \boldsymbol{U} = \mathbf{1} \quad \text{and} \quad \boldsymbol{U}^t = \boldsymbol{U}^{-1} \tag{1.35}$$

with $\det \boldsymbol{U} = 1$ for proper orthonormal transformations. Consequently each unitary transformation is also a similarity transformation, $\tilde{\boldsymbol{\sigma}} \sim \boldsymbol{\sigma}$, where

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{U}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{U} = \boldsymbol{U}^t \cdot \boldsymbol{\sigma} \cdot \boldsymbol{U} \tag{1.36}$$

but not vice versa.

Note: Unitary equivalence distinguishes second order tensors from matrices of order three, since it preserves the length of the tensor $\boldsymbol{\sigma}$ in any coordinate system. In other terms, a zero length tensor in one coordinate system will have zero length in any other coordinate system.

**(k) Cayley-Hamilton Theorem**:

This theorem states that every square matrix satisfies its own characteristic equation. In other terms the scalar polynomial $p(\lambda) = \det(\lambda \mathbf{1} - \boldsymbol{\sigma}^t)$ also holds for the matrix polynomial $p(\boldsymbol{\sigma})$. One important application of the Cayley-Hamilton theorem is to express powers of the matrix $\boldsymbol{\sigma}^k$ as linear combination of $\mathbf{1}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^2$ for $k \geq 3$. In short, the tensor satisfies its characteristic equation.

$$p(\lambda) = \boldsymbol{\sigma}^3 - I_1 \boldsymbol{\sigma}^2 - I_2 \boldsymbol{\sigma} - I_3 = 0 \tag{1.37}$$

**(l) Examples of Unitary (Proper Orthonormal) Transformations:**

1. Plane Rotation:

   $\boldsymbol{U}(\theta, i, j)$ is the identity matrix where the $ii, jj$ entries are replaced by $\cos\theta$ and the $ij$ entry by $-\sin\theta$ and $ji$ by $\sin\theta$.

$$\boldsymbol{U}(\theta, i, j) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{1.38}$$

   It is apparent that this corresponds to the plane coordinate rotation discussed earlier on, where $\boldsymbol{Q} = \boldsymbol{U}^t$.

   Note: Successive application of plane rotations reduces $\boldsymbol{\sigma}$ to diagonal form (Jacobi and Givens method to extract eigenvalues).

2. Householder Transformation:

   $\boldsymbol{U}_w = \mathbf{1} - t\boldsymbol{w}\boldsymbol{w}^t$ is a rank one update of the unit matrix in the form of a reflection such that

$$\boldsymbol{U}_w = \mathbf{1} - 2\boldsymbol{w}\boldsymbol{w}^t \quad \text{with} \quad t = 2(\boldsymbol{w}^t\boldsymbol{w})^{-1} \quad \text{and} \quad \boldsymbol{w}^t\boldsymbol{w} = 1 \tag{1.39}$$

   Consequently, the Householder transformation acts as identity transformation, $\boldsymbol{U}_w\boldsymbol{x} = \boldsymbol{x}$ if $\boldsymbol{x} \perp \boldsymbol{w}$ and $\boldsymbol{U}_w\boldsymbol{w} = -\boldsymbol{w}$ is a reflection.

   Note: Successive application of Householder transformations reduces $\boldsymbol{\sigma}$ to Hessenberg form (in case of symmetry to a tri-diagonal form).

# Chapter 2

# Kinematics of Continua

**Abstract**

In this Chapter, the motion and deformation of continua will be reviewed:

- Kinematics of Motion : $\mathbf{X}, \mathbf{x}, \mathbf{u}$.
- Kinematics of Deformation: $\mathbf{F}, \mathbf{E}, \mathbf{e}$.

## 2.1 Kinematics of Motion:

The motion of continua may be described in two ways:

(a) In the Lagrangean description the continuous medium is considered to be comprised of particles the motion of which is of primary interest. The Lagrangean coordinates describe the spatial variation of a field variable in terms of the particle position $\mathbf{X}$ in the initial reference configuration.

(b) In the Eulerian description the location is of primary interest which is occupied by particles at the time t. The Eulerian coordinates describe the spatial variation of a field variable in terms of the spatial domain $\mathbf{x}$ occupied by the continuum.

Usually, Eulerian coordinates are used to study the motion of fluids through a fixed spatial domain, while Lagrangean coordinates are primarily used to follow the particle motion of solids. In sum, every function expressed in Lagrangean coordinates may be transformed into Eulerian coordinates and vice-versa.

- In Lagrangean coordinates, which are known as material coordinates, the initial position of a particle is used to label the material particle under consideration:
  - Lagrange Coordinates : (Material Description)

$$\mathbf{X} = X_A \mathbf{e}_A \quad \text{where} \quad A = 1, 2, 3 \tag{2.1}$$

where $\mathbf{e}_A$ denote the orthogonal material base vectors.

- In Eulerian coordinates, which are known as spatial coordinates, the location is used to label the material position of a material particle at the time t:
  - Euler Coordinates : (Spatial Description)

$$\mathbf{x} = x_i \mathbf{e}_i \quad \text{where} \quad i = 1, 2, 3 \tag{2.2}$$

  where $\mathbf{e}_i$ denote the orthogonal spatial base vectors.

  The scalar temperature field may be represented by :

  - Lagrangean Coordinates : $T = T(\mathbf{X}, t)$ - material description.
  - Eulerian Coordinates : $T = T(\mathbf{x}, t)$ - spatial description.

  In Lagrangean coordinates the temperature at every material point is studied, while in Eulerian coordinates the temperature at a fixed location which the material occupies is of primary interest.

During the motion, the deformation gradient is defined as:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \Rightarrow d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} \tag{2.3}$$

where $\mathbf{X}$ denotes the position of a arbitrary material particle in the (initial) reference configuration, and $\mathbf{x}$ its position in the current configuration. The determinant $det\mathbf{F}$ is known as the Jacobian of the deformation gradient. Therefore the mapping of the material line element $d\mathbf{X}$ from the reference into the current configuration is defined by :

$$\begin{bmatrix} dx_i \\ dx_j \\ dx_k \end{bmatrix} = \begin{bmatrix} \frac{dx_i}{dX_A} & \frac{dx_i}{dX_B} & \frac{dx_i}{dX_C} \\ \frac{dx_j}{dX_A} & \frac{dx_j}{dX_B} & \frac{dx_j}{dX_C} \\ \frac{dx_k}{dX_A} & \frac{dx_k}{dX_B} & \frac{dx_k}{dX_C} \end{bmatrix} \begin{bmatrix} dX_A \\ dX_B \\ dX_C \end{bmatrix} \tag{2.4}$$

Restriction: the Jacobian $J = \frac{dv}{dV} = \det \mathbf{F}$ measures the volume change, whereby for rigid body motions, $J = 1$ in the absence of deformation. For a one-to-one deformation map the Jacobian must be non-zero and in fact positive if material interpenetration is not permitted, i.e.$\det \mathbf{F} > 0$ must hold. Figure 2.1 illustrates the central role of the deformation gradient in the map.

## 2.2 Polar Decomposition:

The polar decomposition theorem states that the deformation gradient $\mathbf{F}$ may be decomposed uniquely into a positive definite tensor and a proper orthogonal tensor, i.e. the *right* $\mathbf{U}$ or *left* $\mathbf{V}$ *stretch* tensor plus the *rotation* $\mathbf{R}$ tensor.

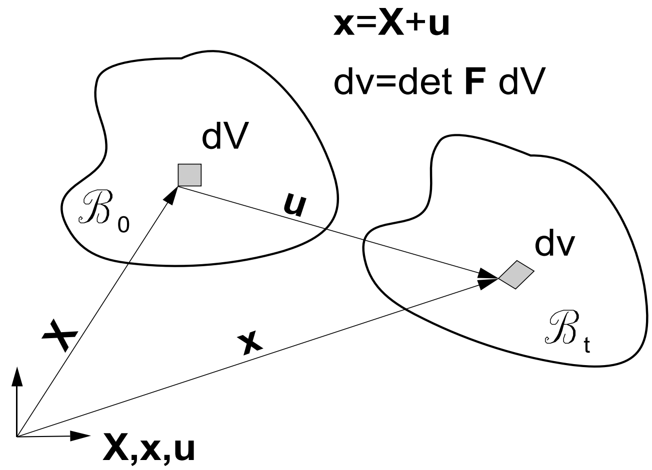


Figure 2.1: The deformation gradient: $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \mathbf{1}$

1. Right Polar Decomposition: $\boxed{\mathbf{F} = \mathbf{R} \cdot \mathbf{U}}$ where: $\det \mathbf{R} = 1$, with $\mathbf{R}^t \cdot \mathbf{R} = \mathbf{1}$, and $\mathbf{R} \cdot \mathbf{R}^t = \mathbf{1}$,

   Note, $\mathbf{U} = \mathbf{U}^t$ such that $\lambda_U > 0$ and

   $$\mathbf{U}^2 = \mathbf{U}^t \cdot \mathbf{R}^t \cdot \mathbf{R} \cdot \mathbf{U} = \mathbf{F}^t \cdot \mathbf{F} \tag{2.5}$$

   Figure 2.2 depicts the sequence of rotating and stretching the line element from the undeformed into the deformed configuration.

2. Left Polar Decomposition: $\boxed{\mathbf{F} = \mathbf{V} \cdot \mathbf{R}}$

   where:

   $$\mathbf{V}^2 = \mathbf{F} \cdot \mathbf{F}^t = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{R}^t \cdot \mathbf{V} \tag{2.6}$$

   Note $\mathbf{V} = \mathbf{V}^t$ such that $\lambda_V > 0$ and $\lambda_V = \lambda_U$

The Physical Meaning of **U** and **V** is:

- Right Decomposition: $d\mathbf{x} = \mathbf{R} \cdot \mathbf{U} \cdot d\mathbf{X}$, which means that $d\mathbf{X}$ is first stretched and then rotated.

- Left Decomposition: $d\mathbf{x} = \mathbf{V} \cdot \mathbf{R} \cdot d\mathbf{X}$, which means that $d\mathbf{X}$ is first rotated and then stretched.

If **F** is non-singular $\Rightarrow \det \mathbf{F} \neq 0$ and positive, then there exists a *unique* decomposition into a proper orthogonal tensor **R** and a *positive definite* tensor **U** or **V**.
Figure 2.3 depicts the sequence of stretching and rotating the line element from the undeformed into the deformed configuration.

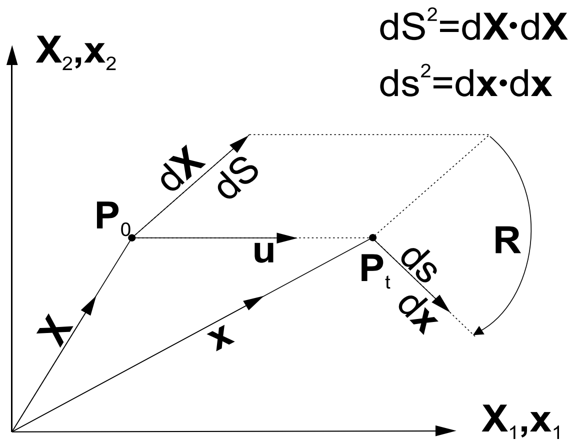


Figure 2.2: Right polar decomposition into RBR and stretch: $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$

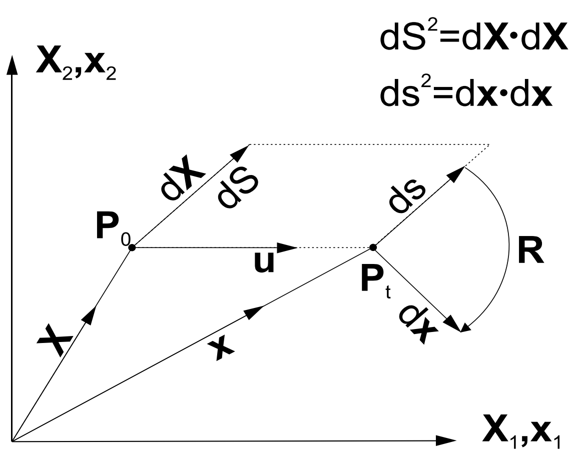


Figure 2.3: Left polar decomposition into stretch and RBR: $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$

## 2.3 Strain Definition:

In this section alternative strain measures are introduced to define strain.

Extensional Engineering Strain:

In the uniaxial case the engineering strain is simply the length change normalized by the original length:

$$\epsilon_{eng} = \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0}. \tag{2.7}$$

The triaxial extension of the engineering strain will be dicussed later on in the context of infinitesimal deformations.

Logarithmic Strain [Hencky, 1928]:

In the uniaxial case the logarithmic strain is defined by integrating the stretch rate:

$$\epsilon_{ln} = \int_{L_0}^{L} \frac{d\ell}{\ell} = ln\frac{L}{L_0} = ln\lambda \quad \text{where} \quad \lambda = \frac{L}{L_0}. \tag{2.8}$$

The Triaxial extension of the logarithmic strain may be described in two ways:

- Lagrangean format : $\boldsymbol{\epsilon}_{ln}^{U} = ln\mathbf{U}$, with principal coordinates which are defined by $\mathbf{e}_U$
- Eulerian format : $\boldsymbol{\epsilon}_{ln}^{V} = ln\mathbf{V}$, with principal Coordinates which are defined by $\mathbf{e}_V$ where the two base triads are related by the rotation $\mathbf{e}_V = \mathbf{R} \cdot \mathbf{e}_U$.

Spectral Representation in terms of principal values and principal directions. :

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \, \mathbf{e}_U^i \otimes \mathbf{e}_U^i \quad \text{and} \quad \boldsymbol{\epsilon}_{ln}^{U} = \sum_{i=1}^{3} (ln\lambda_i) \, \mathbf{e}_U^i \otimes \mathbf{e}_U^i \tag{2.9}$$

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \, \mathbf{e}_V^i \otimes \mathbf{e}_V^i \quad \text{and} \quad \boldsymbol{\epsilon}_{ln}^{V} = \sum_{i=1}^{3} (ln\lambda_i) \, \mathbf{e}_V^i \otimes \mathbf{e}_V^i \tag{2.10}$$

### 2.3.1 Lagrangean Strain Measures:

If $dS$ is the infinitesimal line element measuring the initial distance of two points, and if $ds$ is the current length of the line element between those two points, then one can define the extensional deformation is defined as:

$$ds^2 - dS^2 = d\mathbf{x}^t \cdot d\mathbf{x} - d\mathbf{X}^t \cdot d\mathbf{X} = d\mathbf{X}^t \cdot [\mathbf{F}^t \cdot \mathbf{F} - \mathbf{1}] \cdot d\mathbf{X} \tag{2.11}$$

The term in bracket defines the Green strain which is related to the right stretch tensor by:

$$\boxed{\mathbf{E}_G = \tfrac{1}{2}\left[\mathbf{F}^t \cdot \mathbf{F} - \mathbf{1}\right] = \tfrac{1}{2}\left[\mathbf{U}^2 - \mathbf{1}\right]} \tag{2.12}$$

In indicial form,

$$\mathbf{F} = \frac{\partial x_i}{\partial X_A}\,\mathbf{e}_i \otimes \mathbf{e}_A \tag{2.13}$$

$$\mathbf{U}^2 = \frac{\partial x_i}{\partial X_A} \cdot \frac{\partial x_i}{\partial X_B} \tag{2.14}$$

$$E^G_{AB} = \frac{1}{2}\left(\frac{\partial x_i}{\partial X_A} \cdot \frac{\partial x_i}{\partial X_B} - \delta_{AB}\right) \tag{2.15}$$

**Generalized Lagrangean Strain** [Doyle-Erickson 1956]:

The generalized format of Lagrangean strain is defined as:

$$\boxed{\mathbf{E}_m = \tfrac{1}{m}\left[\mathbf{U}^m - \mathbf{1}\right]} \tag{2.16}$$

where m is an integer, where

m=0, defines the logarithmic strain:

$$\boxed{\mathbf{E}_0 = ln\mathbf{U}} \tag{2.17}$$

m=1, defines the Biot strain:

$$\boxed{\mathbf{E}_1 = \left[\mathbf{U} - \mathbf{1}\right]} \tag{2.18}$$

m=2, defines the Green strain:

$$\boxed{\mathbf{E}_2 = \tfrac{1}{2}\left[\mathbf{U}^2 - \mathbf{1}\right]} \tag{2.19}$$

Using the definition of stretch $\lambda = \frac{L}{L_0}$, then 1-dim examples of extensional strain measures include :

$$\begin{aligned} E_0 &= ln\lambda = ln\tfrac{L}{L_0} \\ E_1 &= \lambda - 1 = \tfrac{L-L_0}{L_0} \\ E_2 &= \tfrac{1}{2}\left[\lambda^2 - 1\right] = \tfrac{1}{2}\left[\tfrac{L^2-L_0^2}{L_0^2}\right] \end{aligned} \tag{2.20}$$

We see that $E_0$ reduces to the logarithmic Hencky strain, $E_1$ to the engineering strain, and $E_2$ to the Green strain.

**Strain-Displacement Relationship**

If $\mathbf{X}$ and $\mathbf{x}$ are the initial and current position vectors of a particle, then the displacement vector defines the relation between $\mathbf{X}$ and $\mathbf{x}$ as:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) = \mathbf{u}(\mathbf{X}, t) + \mathbf{X} \tag{2.21}$$

The deformation gradient is related to the displacement gradient,

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \mathbf{1} = \mathbf{H} + \mathbf{1} \quad \text{where} \quad \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{H} \tag{2.22}$$

With $\mathbf{F} = \mathbf{H} + \mathbf{1}$ the material stretch tensor reads,

$$\mathbf{U}^2 = \mathbf{F}^t \cdot \mathbf{F} = \mathbf{1} + \mathbf{H} + \mathbf{H}^t + \mathbf{H}^t \cdot \mathbf{H} \tag{2.23}$$

and the Lagrangean strain-displacement relationship defines the Green strain in terms of:

$$\boxed{\mathbf{E}_G = \tfrac{1}{2}[\mathbf{1} + \mathbf{H} + \mathbf{H}^t + \mathbf{H}^t \cdot \mathbf{H}]} \tag{2.24}$$

### 2.3.2 Eulerian Strain Measures :

Parallel to the material description, the extensional deformation may be defined in terms of spatial coordinates,

$$ds^2 - dS^2 = d\mathbf{x}^t \cdot [\mathbf{1} - \mathbf{F}^{-t} \cdot \mathbf{F}^{-1}] \cdot d\mathbf{x} \tag{2.25}$$

The term in the bracket defines the Almansi strain which is related to the left stretch tensor by,

$$\boxed{\mathbf{e}_A = \tfrac{1}{2}\left[\mathbf{1} - \mathbf{F}^{-t} \cdot \mathbf{F}^{-1}\right] = \tfrac{1}{2}\left[\mathbf{1} - (\mathbf{V}^2)^{-1}\right]} \tag{2.26}$$

**Generalized Eulerian Strain** [Doyle-Erickson 1956]:

The generalized Eulerian strain tensor is defined as:

$$\boxed{\mathbf{e}_m = \tfrac{1}{m}\left(\mathbf{V}^m - \mathbf{1}\right)} \tag{2.27}$$

where m is an integer.

m=0, defines the spatial format of the logarithmic strain:

$$\boxed{\mathbf{e}_0 = ln\mathbf{V}} \tag{2.28}$$

m=-1, defines the spatial format of the Biot strain

$$\boxed{\mathbf{e}_{-1} = [\mathbf{1} - \mathbf{V}^{-1}]} \tag{2.29}$$

m=-2, defines the Almansi strain

$$\boxed{\mathbf{e}_{-2} = \tfrac{1}{2}\left[\mathbf{1} - \mathbf{V}^{-2}\right]} \tag{2.30}$$

using the inverse stretch measure $\frac{1}{\lambda} = \frac{L_0}{L}$, then 1-dim examples include:

$$\begin{aligned} e_0 &= -ln\tfrac{1}{\lambda} = ln\tfrac{L}{L_0} \\ e_{-1} &= 1 - \tfrac{1}{\lambda} = \tfrac{L-L_0}{L} \\ e_{-2} &= \tfrac{1}{2}\left[1 - \tfrac{1}{\lambda^2}\right] = \tfrac{1}{2}\left[\tfrac{L^2-L_0^2}{L^2}\right] \end{aligned} \tag{2.31}$$

We see that in 1-dim $e_0$ coincides with the logarithmic Hencky strain, $e_1$ corresponds to the engineering strain in which the length change is however normalized by the current length, while the Almansi strain $e_{-2}$ corresponds to the Green strain $E_2$.

**Strain-Displacement Relationship**

In the spatial description the motion is described by the inverse relation:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t) \tag{2.32}$$

The spatial displacement gradient is defined as:

$$\mathbf{h} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{1} - \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \mathbf{1} - \mathbf{F}^{-1} \tag{2.33}$$

or

$$\mathbf{F}^{-1} = \mathbf{1} - \mathbf{h} \tag{2.34}$$

$$\mathbf{V}^{-2} = \mathbf{F}^{-t} \cdot \mathbf{F}^{-1} = \mathbf{1} - \mathbf{h} - \mathbf{h}^t + \mathbf{h}^t \cdot \mathbf{h} \tag{2.35}$$

Substituting into the expression of the Almansi strain $\mathbf{e}_A$ renders the Eulerian strain-displacement relationship in the form,

$$\boxed{\mathbf{e}_A = \tfrac{1}{2}[\mathbf{h} + \mathbf{h}^t - \mathbf{h}^t \cdot \mathbf{h}]} \tag{2.36}$$

### 2.3.3 Infinitesimal Deformations and Rotations:

If the displacement gradient $\mathbf{H}$ is very small, then $\det \mathbf{H} << 1 \Rightarrow \det(\mathbf{H}^t \cdot \mathbf{H}) \simeq 0$. Thus,

$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^t\right] \simeq \frac{1}{2}\left[\mathbf{h} + \mathbf{h}^t\right] \tag{2.37}$$

In short, in the case of *"Infinitesimal Deformations"* the spatial and the material displacement gradients coincide,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} \sim \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \tag{2.38}$$

Additive decomposition into symmetric and skew-symmetric components leads to

$$\mathbf{H} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \frac{1}{2}[\mathbf{H} + \mathbf{H}^t] + \frac{1}{2}[\mathbf{H} - \mathbf{H}^t] \tag{2.39}$$

where the symmetric part defines the traditional linearized strain tensor

$$\epsilon_{ij} = \frac{1}{2}[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}] \tag{2.40}$$

and where the skew-symmetric part defines the infinitesimal rotation tensor

$$\omega_{ij} = \frac{1}{2}[\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}] \tag{2.41}$$

such that $\epsilon_{ij} = \epsilon_{ji}$ and $\omega_{ij} = -\omega_{ji}$.

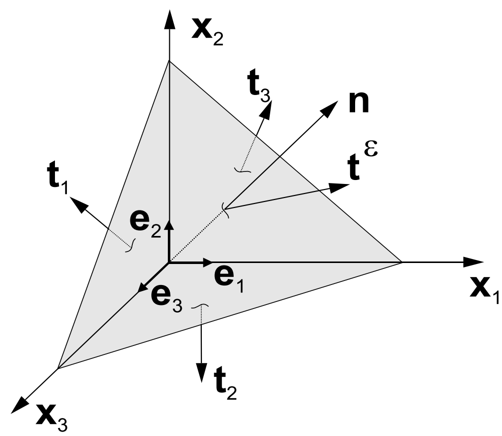


Figure 2.4: Traction vector of strain tensor on normal plane: $\mathbf{t}^{\epsilon} = \boldsymbol{\epsilon}^t \cdot \mathbf{n}$

In analogy to stress we can identify the traction vector of strain $\mathbf{t}^{\epsilon}$ as linear mapping of the strain tensor $\boldsymbol{\epsilon}$ onto the surface with the normal $\mathbf{n}$.

$$\mathbf{t}^{\epsilon} = \boldsymbol{\epsilon}^t \cdot \mathbf{n} \quad \text{or} \quad t_i^{\epsilon} = \epsilon_{ji}\, n_j \tag{2.42}$$

This transformation may be viewed as a projection of the strain tensor onto the plane with the normal vector $\mathbf{n}^t = [n_1, n_2, n_3]$, where

$$\boldsymbol{t}^{\epsilon} = \boldsymbol{\epsilon}^t \cdot \boldsymbol{n} = \begin{bmatrix} \epsilon_{11}n_1 + \epsilon_{21}n_2 + \epsilon_{31}n_3 \\ \epsilon_{12}n_1 + \epsilon_{22}n_2 + \epsilon_{32}n_3 \\ \epsilon_{13}n_1 + \epsilon_{23}n_2 + \epsilon_{33}n_3 \end{bmatrix} \tag{2.43}$$

Figure 2.4 illustrates the traction vector projecting the strain tensor onto the surface with the normal $\boldsymbol{n}$. Thereby, the strain tensor is uniquely defined in terms of the traction vectors on three mutually orthogonal planes:

$$\boldsymbol{\epsilon} = [\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \tag{2.44}$$

with the understanding that $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^t$ is symmetric. In this notation the first subscript $i$ in $[\epsilon_{ij}]$ refers to normal direction of the *area* element and the second subscript $j$ refers to the direction of action of the traction component.

Figure 2.5 illustrates the components of the strain tensor which act on the surface of the infinitesimal cube aligned with the cartesian coordinates.

The traction vector $\mathbf{t}^{\epsilon}$ may be decomposed into normal and tangential components on the plane $\mathbf{n}$,

$$\epsilon_n = \mathbf{n} \cdot \mathbf{t}^{\epsilon} = \mathbf{n} \cdot \boldsymbol{\epsilon}^t \cdot \mathbf{n} \tag{2.45}$$

and

$$(\tau_n)^2 = |\mathbf{t}^{\epsilon}|^2 - \epsilon_n^2 = \mathbf{t}^{\epsilon} \cdot \mathbf{t}^{\epsilon} - (\mathbf{n} \cdot \mathbf{t}^{\epsilon})^2 \tag{2.46}$$

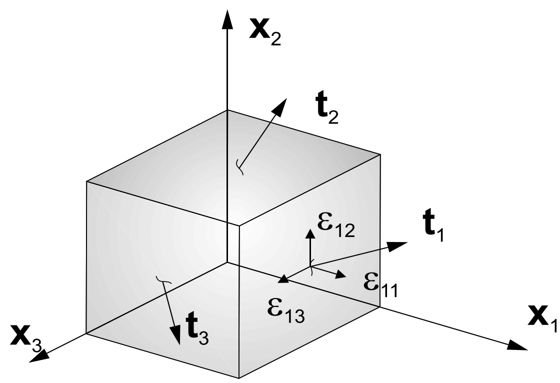


Figure 2.5: Cartesian components of strain tensor

which are the normal strain and the tangential strain components on that plane.

**Principal Strain Coordinates:**

There exist a nonzero vector $\mathbf{n}_p$ such that the linear transformation $\boldsymbol{\epsilon}^t \cdot \mathbf{n}_p$ is a multiple of $\mathbf{n}_p$. In this case $\mathbf{n} \| \mathbf{t}^\epsilon$ we talk about principal direction $\mathbf{n} = \mathbf{n}_p$ in which the tangential shear components vanish,

$$\boldsymbol{\epsilon}^t \cdot \mathbf{n}_p = \lambda \mathbf{n}_p \tag{2.47}$$

The eigenvalue problem is equivalent to stating that

$$[\boldsymbol{\epsilon}^t - \lambda \mathbf{1}] \cdot \mathbf{n}_p = \mathbf{0} \tag{2.48}$$

For a non-trivial solution $\mathbf{t}_p \neq \mathbf{0}$, the matrix $[\boldsymbol{\epsilon}^t - \lambda \mathbf{1}]$ must be singular. Consequently, $\det(\boldsymbol{\epsilon}^t - \lambda \mathbf{1}) = 0$ generates the characteristic polynomial

$$p(\lambda) = \det(\boldsymbol{\epsilon}^t - \lambda \mathbf{1}) = \lambda^3 - I_1^\epsilon \lambda^2 - I_2^\epsilon \lambda - I_3^\epsilon = 0 \tag{2.49}$$

where the three principal strain invariants are,

$$I_1^\epsilon = \mathrm{tr}\boldsymbol{\epsilon} = \epsilon_{ii} = \epsilon_1 + \epsilon_2 + \epsilon_3 \tag{2.50}$$

$$I_2^\epsilon = \frac{1}{2}[\mathrm{tr}\boldsymbol{\epsilon}^2 - (\mathrm{tr}\boldsymbol{\epsilon})^2] = \frac{1}{2}[\epsilon_{ij}\epsilon_{ij} - \epsilon_{ii}^2] = -[\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1] \tag{2.51}$$

$$I_3^\epsilon = \det \boldsymbol{\epsilon} = \epsilon_1\epsilon_2\epsilon_3 \tag{2.52}$$

According to the fundamental theorem of algebra, a polynomial of degree 3 has exactly 3 roots, thus each matrix $\boldsymbol{\epsilon} \in \Re^3$ has 3 eigenvalues $\lambda_1 = \epsilon_1, \lambda_2 = \epsilon_2, \lambda_3 = \epsilon_3$.
Note: all three eigenvalues are real since the strain tensor $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^t$ is symmetric.

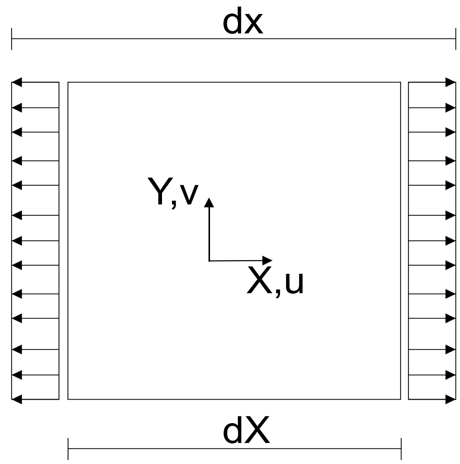


Figure 2.6: Infinitesimal normal strain: $\epsilon_x = \frac{\partial u}{\partial x}$

### 2.3.4 2-D State of Plane Strain:

For illustration Figures 2.6 - 2.9 depict the individual components of the infinitesimal strain and rotation tensor for 2-D motions and in plane strain. In the case of 'small' displacement gradients we do no longer need to distinguish between the derivatives with regard to deformed and undeformed configurations. Hence the symmetric part of the displacement gradient renders the linearized tensor of engineering strain,

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_x & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_y \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & 0.5[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}] \\ 0.5[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}] & \frac{\partial v}{\partial y} \end{bmatrix} \tag{2.53}$$

where $\epsilon_{xy} = \epsilon_{yx} = 0.5\gamma_{xy}$. The skew-symmetric part of the displacement gradient renders the linearized tensor of average rotation,

$$[\omega_{ij}] = \begin{bmatrix} 0 & \omega_{xy} \\ \omega_{yx} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0 \end{bmatrix} \tag{2.54}$$

where $\omega_{xy} = -\omega_{yx}$.
Coordinate transformation of the base vectors by the unitary z-rotation by the angle $\theta$,

$$\mathbf{Q} = [Q_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2.55}$$

results in the tensor transformation of the second order strain tensor. The rotated components read,

$$\tilde{\boldsymbol{\epsilon}} = \mathbf{Q} \cdot \boldsymbol{\epsilon} \cdot \mathbf{Q}^t = \begin{bmatrix} \tilde{\epsilon}_x & \tilde{\epsilon}_{xy} \\ \tilde{\epsilon}_{yx} & \tilde{\epsilon}_y \end{bmatrix} \tag{2.56}$$

E.g. the normal strain in the rotated coordinate system $\tilde{\epsilon}_x$ reduces to the traditional transformation expression,

$$\tilde{\epsilon_x} = \epsilon_x \cos^2\theta + \epsilon \sin^2\theta + 2\,\epsilon_{xy} \sin\theta \cos\theta \tag{2.57}$$

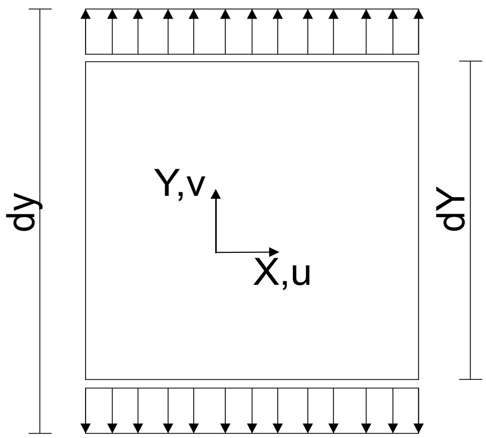


Figure 2.7: Infinitesimal normal strain: $\epsilon_y = \frac{\partial v}{\partial y}$

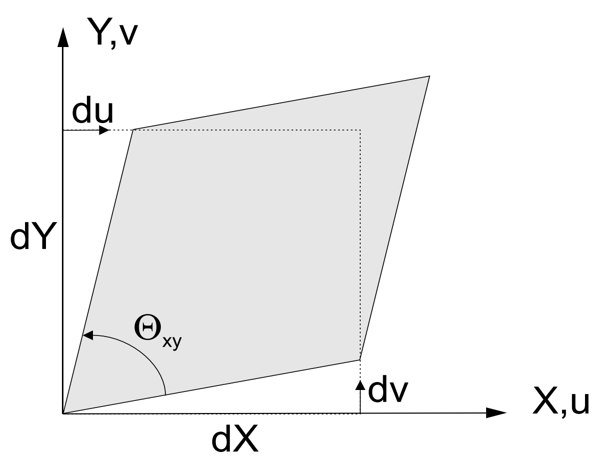


Figure 2.8: Infinitesimal shear strain: $\epsilon_{xy} = \frac{1}{2}[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}]$

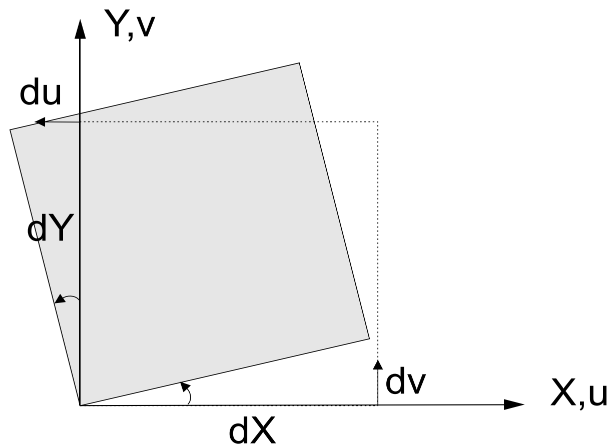


Figure 2.9: Infinitesimal rigid body rotation: $\omega_{xy} = \frac{1}{2}[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}]$

and in terms of double angles,

$$\tilde{\epsilon_x} \;=\; \frac{1}{2}[\epsilon_x + \epsilon_y] + \frac{1}{2}[\epsilon_x - \epsilon_y]cos2\theta + \epsilon_{xy}sin2\theta \tag{2.58}$$

In analogy, the shear strain in the rotated coordinate system $\tilde{\epsilon}_{xy}$ reads,

$$\tilde{\epsilon}_{xy} \;=\; -\epsilon_x \sin\theta\cos\theta + \epsilon_y \sin\theta\cos\theta + \epsilon_{xy}[\cos^2\theta - \sin^2\theta] \tag{2.59}$$

and in terms of double angles,

$$\tilde{\epsilon}_{xy} \;=\; -\frac{1}{2}[\epsilon_x - \epsilon_y]sin2\theta + \epsilon_{xy}cos2\theta \tag{2.60}$$

**Principal Values**

There is one coordinate system along which the shear deformations vanish which distinguishes the principal coordinates from all others. Eigenvalue analysis of the strain tensor,

$$\begin{bmatrix} \epsilon_x & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_y \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} \;=\; \lambda \begin{bmatrix} n_x \\ n_y \end{bmatrix} \tag{2.61}$$

renders the characteristic polynomial,

$$p(\lambda) = \det(\boldsymbol{\epsilon} - \lambda \mathbf{1}) = \lambda^2 - [\epsilon_x + \epsilon_y]\lambda + \epsilon_x\epsilon_y - \epsilon_{xy}\epsilon_{yx} = 0 \tag{2.62}$$

Given $\epsilon_{yx} = \epsilon_{xy} = 0.5\gamma_{xy}$, the two roots of the second order polynomial are the principal strains,

$$\lambda_{1,2} = \frac{1}{2}[\epsilon_x + \epsilon_y] \pm \sqrt{\frac{1}{4}[\epsilon_x - \epsilon_y]^2 + \epsilon_{xy}^2} \tag{2.63}$$

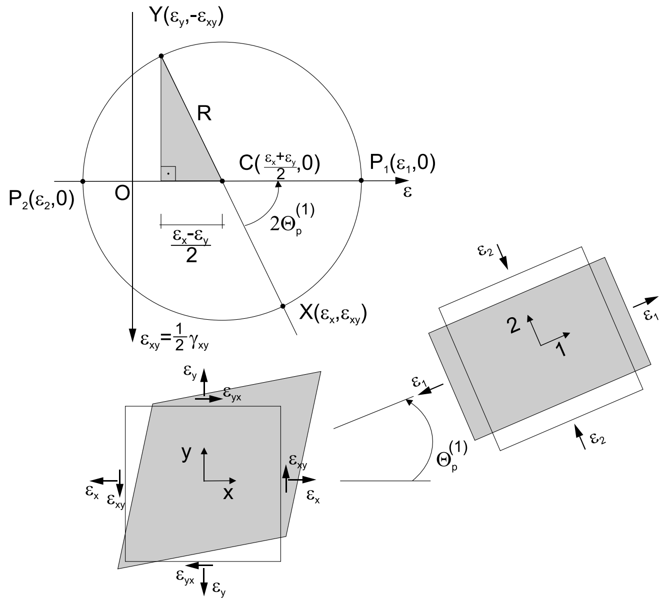


Figure 2.10: Mohr Circle of Strain: $\epsilon_{1,2} = \epsilon_{avg} \pm R \quad \text{and} \quad \gamma_{max} = 2\,R$

A very ingenious geometric illustration of the coordinate transformation dates back to Otto Mohr [1887]. According to the geometric construction of the Mohr circle the principal normal strains and the maximum shear strain shown in Figure 2.10 are:

$$\epsilon_{1,2} = \epsilon_{avg} \pm R \quad \text{and} \quad \gamma_{max} = 2\,R \tag{2.64}$$

The average normal strain, $\epsilon_{avg} = \frac{1}{2}[\epsilon_x + \epsilon_y]$ locates the center of Mohr's circle, and the radius $R = \sqrt{\frac{1}{4}[\epsilon_x - \epsilon_y]^2 + \epsilon_{xy}^2}$ defines the locus of all possible representations of the 2-D state of strain.
Thereby, the major principal strain axis is inclined with the x-axes by the angle,

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} \tag{2.65}$$

which coincides with the direction of the major eigenvector $\mathbf{n}_p^{(1)}$.

# Chapter 3

# Stress and Differential Equations of Motion:

The balance laws of continuum mechanics comprise statements as follows:

1. Balance of Linear Momentum.
2. Balance of Angular Momentum.
3. Balance of Mass.
4. Balance of Energy (First Law of Thermodynamics).

In this chapter we focus on linear and angular momentum and the local (differential ) statements of equilibrium on a infinitesimal surface and volume element.

## 3.1 The Cauchy Lemma:

This lemma states pointwise balance of tractions across any surface in the interior of the body.

$$\mathbf{t}(\mathbf{x},+\mathbf{n}) \;+\; \mathbf{t}(\mathbf{x},-\mathbf{n}) \;=\; \mathbf{0} \tag{3.1}$$

In other terms, the traction vector must be equal and pointing in the opposite directions along an infinitesimal surface element of arbitrary orientation.

$$\boxed{\mathbf{t}(\mathbf{x},+\mathbf{n}) \;=\; -\mathbf{t}_n(\mathbf{x},-\mathbf{n})} \tag{3.2}$$

This is continuum statement is equivalent to Newton's third law for particles, that action equals reaction. It forms the basis of free body diagrams used in statics for exposing internal forces and stress resultants.
Figure 3.1 illustrates the continuity of the traction vector across an infinitesimal line element. The underlying equilibrium argument simply states that traction vector must be equal and opposite at the opposite sides of the imaginary surface.

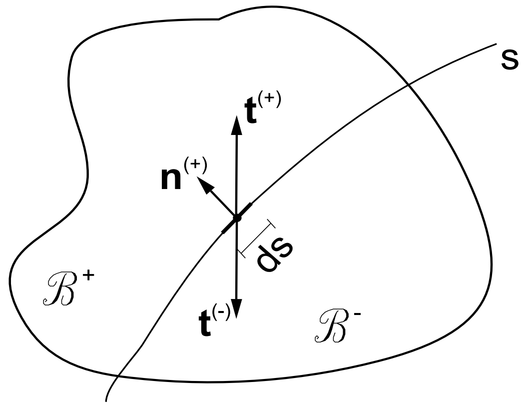


Figure 3.1: Traction equilibrium across arbitrary surface element

## 3.2 The Cauchy Theorem:

The traction vector $\mathbf{t}$ is a linear mapping of the stress tensor $\boldsymbol{\sigma}$ onto the surface with the normal $\mathbf{n}$.

$$\mathbf{t} \;=\; \boldsymbol{\sigma}^t \cdot \mathbf{n} \quad \text{or in index notation} \quad t_i \;=\; \sigma_{ji}\, n_j \tag{3.3}$$

Considering equilibrium of an elementary tetrahedron we find:

$$\mathbf{t} \;=\; \mathbf{n}_1\, \mathbf{t}_1 \;+\; \mathbf{n}_2\, \mathbf{t}_2 \;+\; \mathbf{n}_3\, \mathbf{t}_3 \tag{3.4}$$

Figure 3.2 illustrates the traction vector projecting the stress tensor onto the surface with the normal $\boldsymbol{n}$. The underlying equilibrium argument infers that the traction acting on any surface element is uniquely defined by the stress tensor. In terms of cartesian coordinates the Cauchy stress tensor $\boldsymbol{\sigma}^t$ is comprised of the components of the traction vectors on three mutually orthogonal planes:

$$\boldsymbol{\sigma} \;=\; [\sigma_{ij}] \;=\; \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{3.5}$$

In this notation the first subscript $i$ in $[\sigma_{ij}]$ refers to normal direction of the *area* element and the second subscript $j$ refers to the direction of action of the traction component.
Figure 3.3 illustrates the components of the Cauchy-stress tensor which act on the surface of the infinitesimal cube aligned with the cartesian coordinates.
Given the stress tensor, the traction vector is uniquely defined on any arbitrary plane with the normal $\boldsymbol{n}$, the components of which are,

$$\boldsymbol{t} = \boldsymbol{\sigma}^t \cdot \boldsymbol{n} = \begin{bmatrix} \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3 \\ \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{32}n_3 \\ \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3 \end{bmatrix} \tag{3.6}$$

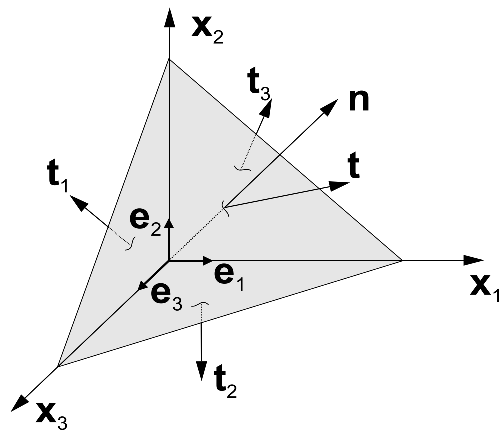


Figure 3.2: Cauchy Theorem: $\mathbf{t} \ = \boldsymbol{\sigma}^t \cdot \mathbf{n}$

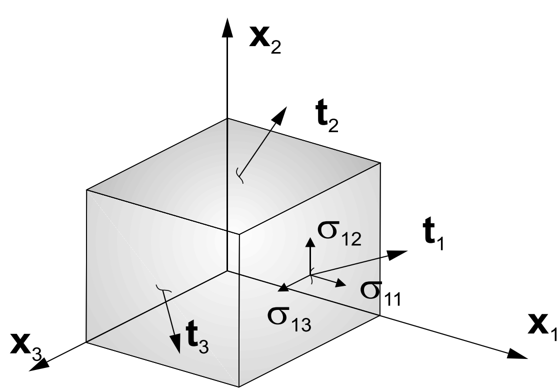


Figure 3.3: Cartesian components of stress tensor

The traction vector $\boldsymbol{t}$ may be decomposed into normal and tangential components on the plane $\boldsymbol{n}$,

$$\sigma_n = \boldsymbol{n} \cdot \boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma}^t \cdot \boldsymbol{n} \tag{3.7}$$

and

$$(\tau_n)^2 = |\boldsymbol{t}|^2 - \sigma_n^2 = \boldsymbol{t} \cdot \boldsymbol{t} - (\boldsymbol{n} \cdot \boldsymbol{t})^2 \tag{3.8}$$

which are the normal stress and the tangential stress components on that plane.

**Eigenvalues and Eigenvectors:**

There exist a nonzero vector $\boldsymbol{n}_p$ such that the linear transformation $\boldsymbol{\sigma}^t \cdot \boldsymbol{n}_p$ is a multiple of $\boldsymbol{n}_p$. In this case $\boldsymbol{n} \| \boldsymbol{t}$ we talk about principal direction $\boldsymbol{n} = \boldsymbol{n}_p$ in which the tangential shear components vanish,

$$\boldsymbol{\sigma}^t \cdot \boldsymbol{n}_p = \lambda \boldsymbol{n}_p \tag{3.9}$$

The eigenvalue problem is equivalent to stating that

$$[\boldsymbol{\sigma}^t - \lambda \mathbf{1}] \cdot \boldsymbol{n}_p = \mathbf{0} \tag{3.10}$$

For a non-trivial solution $\boldsymbol{n}_p \neq \mathbf{0}$ the matrix $(\boldsymbol{\sigma}^t - \lambda \mathbf{1})$ must be singular. Consequently, $\det(\boldsymbol{\sigma}^t - \lambda \mathbf{1}) = 0$ generates the characteristic polynomial

$$p(\lambda) = \det(\boldsymbol{\sigma}^t - \lambda \mathbf{1}) = \lambda^3 - I_1 \lambda^2 - I_2 \lambda - I_3 = 0 \tag{3.11}$$

where the three principal invariants are,

$$I_1 = \mathrm{tr}\boldsymbol{\sigma} = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 \tag{3.12}$$

$$I_2 = \frac{1}{2}[tr\boldsymbol{\sigma}^2 - (\mathrm{tr}\boldsymbol{\sigma})^2] = \frac{1}{2}[\sigma_{ij}\sigma_{ij} - \sigma_{ii}^2] = -[\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1] \tag{3.13}$$

$$I_3 = \det \boldsymbol{\sigma} = \sigma_1 \sigma_2 \sigma_3 \tag{3.14}$$

According to the fundamental theorem of algebra, a polynomial of degree 3 has exactly 3 roots, thus each matrix $\boldsymbol{\sigma} \in \Re^3$ has 3 eigenvalues $\lambda_1 = \sigma_1, \lambda_2 = \sigma_2, \lambda_3 = \sigma_3$. If a polynomial with real coefficients has some non-real complex zeroes, they must occur in conjugate pairs.
Note: all three eigenvalues are real when the stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$ is symmetric.
Further, $\det(\boldsymbol{\sigma}^t - \lambda \mathbf{1}) = (-1)^3 \det(\lambda \mathbf{1} - \boldsymbol{\sigma}^t)$, thus the roots of both characteristic equations are the same.

### 3.2.1 2-D State of Plane Stress:

For illustration we consider the 2-D state of plane stress where the stress tensor is comprised of the non-zero in-plane components,

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \epsilon_y \end{bmatrix} \tag{3.15}$$

Coordinate transformation of the base vectors by the unitary z-rotation by the angle $\theta$,

$$\mathbf{Q} = [Q_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{3.16}$$

results in the tensor transformation of the second order stress tensor. The rotated components read,

$$\tilde{\boldsymbol{\sigma}} = \mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^t \;=\; \begin{bmatrix} \tilde{\sigma}_x & \tilde{\tau}_{xy} \\ \tilde{\tau}_{yx} & \tilde{\sigma}_y \end{bmatrix} \tag{3.17}$$

E.g. the normal stress $\tilde{\sigma}_x$ reduces to the traditional transformation expression,

$$\tilde{\sigma_x} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy}\sin\theta\cos\theta \tag{3.18}$$

and in terms of double angles,

$$\tilde{\sigma_x} \;=\; \frac{1}{2}[\sigma_x+\sigma_y] + \frac{1}{2}[\sigma_x-\sigma_y]cos2\theta + \tau_{xy}sin2\theta \tag{3.19}$$

In analogy, the shear stress reads in the rotated coordinate system,

$$\tilde{\tau}_{xy} = -\sigma_x\sin\theta\cos\theta + \sigma_y\sin\theta\cos\theta + \tau_{xy}[\cos^2\theta - \sin^2\theta] \tag{3.20}$$

and in terms of double angles,

$$\tilde{\tau}_{xy} \;=\; -\frac{1}{2}[\sigma_x-\sigma_y]sin2\theta + \tau_{xy}cos2\theta \tag{3.21}$$

**Principal Values**

There is one coordinate system along which the shear deformations vanish which distinguishes the principal coordinates from all others. Eigenvalue analysis of the 2-D stress tensor,

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}\begin{bmatrix} n_x \\ n_y \end{bmatrix} \;=\; \lambda\begin{bmatrix} n_x \\ n_y \end{bmatrix} \tag{3.22}$$

renders the characteristic polynomial,

$$p(\lambda) = \det(\boldsymbol{\sigma} - \lambda\mathbf{1}) = \lambda^2 - [\sigma_x+\sigma_y]\lambda + \sigma_x\sigma_y - \tau_{xy}\tau_{yx} = 0 \tag{3.23}$$

Given $\tau_{yx} = \tau_{xy}$, the two roots of the second order polynomial are the principal strains,

$$\lambda_{1,2} = \frac{1}{2}[\sigma_x+\sigma_y] \pm \sqrt{\frac{1}{4}[\sigma_x-\sigma_y]^2 + \tau_{xy}^2} \tag{3.24}$$

A very ingenious geometric illustration of the coordinate transformation dates back to Otto Mohr [1887]. According to the geometric construction of the Mohr circle the principal normal strains and the maximum shear strain shown in Figure 2.10 are:

$$\sigma_{1,2} = \sigma_{avg} \pm R \quad \text{and} \quad \tau_{max} = \; R \tag{3.25}$$

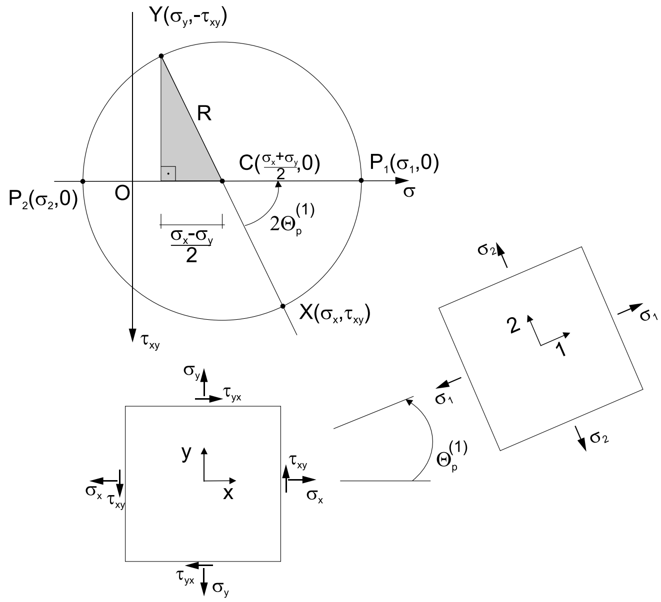


Figure 3.4: Mohr Circle of Strain: $\sigma_{1,2} = \sigma_{avg} \pm R$ and $\tau_{max} = R$

The average normal strain, $\sigma_{avg} = \frac{1}{2}[\sigma_x + \sigma_y]$ locates the center of Mohr's circle, and the radius $R = \sqrt{\frac{1}{4}[\sigma_x - \sigma_y]^2 + \tau_{xy}^2}$ defines the locus of all possible representations of the 2-D state of stress.
Thereby, the major principal strain axis is inclined with the x-axes by the angle,

$$\tan 2\theta_p = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} \tag{3.26}$$

which coincides with the direction of the major eigenvector $\mathbf{n}_p^{(1)}$.

### 3.2.2 Differential Equations of Equilibrium:

For the sake of completeness we consider the differential argument of equilibrium in a 2-D body subject to distributed body forces $b_x, b_y$ per unit area and surface tractions $\tilde{t}_x, \tilde{t}_y$ per unit length. Figure 3.4 shows the physical arguments of differential equilibrium on an infinitesimal area element of the size $da = dxdy$. Considering the 2-D state of plane stress force equilibrium in the x- and y-directions lead to the differential equations below :

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + b_x = 0 \tag{3.27}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0 \tag{3.28}$$

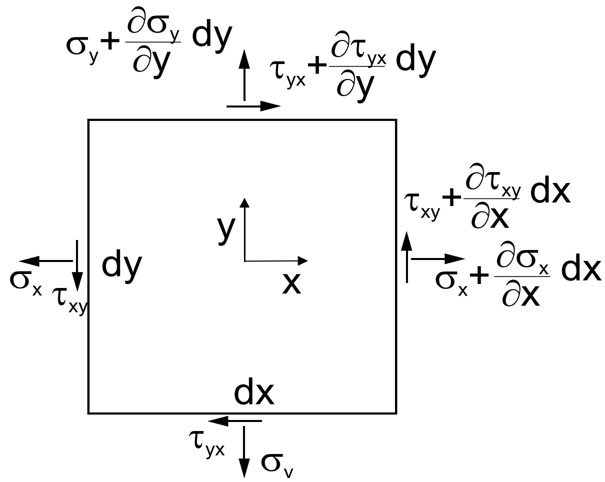


Figure 3.5: Differential equations of equilibrium in the interior area element $da$: $div\boldsymbol{\sigma}^t + \mathbf{b} = \mathbf{0}$

They define equilibrium in terms of Cauchy stress in the interior of the deformed body $\mathcal{B}$ and need to be augmented by moment equilibrium around the out-of-plane z-axis. This leads to the additional argument of conjugate shear stress, where

$$\tau_{xy} = \tau_{yx} \tag{3.29}$$

Considering traction boundary conditions on the surface of the deformed body where $\boldsymbol{\sigma}^t \cdot \mathbf{n} = \tilde{\mathbf{t}}$. Here $\tilde{\mathbf{t}}$ denote prescribed surface tractions in terms of their normal and tangential components acting on the deformed surface area element $ds$.
Figure 3.5 illustrates the boundary conditions at a surface element subjected to prescribed surface tractions $\tilde{\mathbf{t}}$.
The traction boundary conditions read,

$$\tilde{t}_x = \sigma_x n_x + \tau_{yx} n_y \tag{3.30}$$

$$\tilde{t}_y = \tau_{xy} n_x + \sigma_y n_y \tag{3.31}$$

where $n_x = cos\theta$ and $n_y = sin\theta$ denote the direction cosines in terms of the angle $\theta$ in 2-D.

## 3.3 Balance of Linear Momentum:

To generalize the local equilibrium arguments in 2-D let us consider distributed forces acting in the body (per unit volume) and on the surface of the body (per unit surface area):

- $\mathbf{b}$ : body force/unit volume(i.e. self weight, centrifugal forces).
- $\tilde{\mathbf{t}}$ : prescribed surface tractions (pressure, tangential shear).

Integrating the distributed forces over part I of the body $\mathcal{B}$ defines the resultant force vector acting on the body,

$$\mathbf{f} \; = \; \int_{\mathcal{B}} \mathbf{b} \, dv + \int_{\partial\mathcal{B}} \mathbf{t}_n \, da \tag{3.32}$$

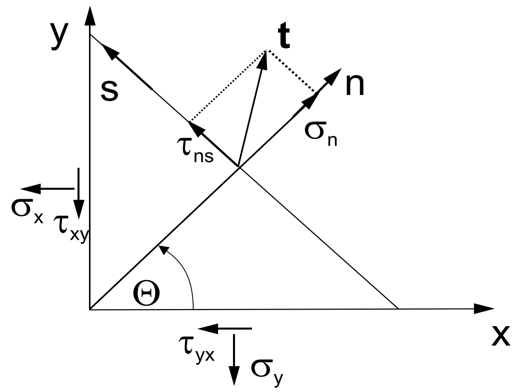


Figure 3.6: Equations of equilibrium at the surface element $ds$: $\boldsymbol{\sigma}^t \cdot \boldsymbol{n} = \tilde{\boldsymbol{t}}$

Note: *"Forces"* are measured only *'indirectly'* through their action on deformable solids.

Linear momentum is defined as:

$$\mathbf{i} = \int_{\mathcal{B}} \rho \, \dot{\mathbf{u}} \, dv \tag{3.33}$$

Application of Newton's second law $\sum \mathbf{f} = \mathbf{m} \cdot \mathbf{a}$ to the control volume of the body renders:

$$\frac{D}{Dt}\mathbf{i} \; = \; \mathbf{f} \tag{3.34}$$

"Dynamic equilibrium" or the balance of linear momentum may be expressed as

$$\int_{\mathcal{B}} \frac{D}{Dt}(\rho\dot{\mathbf{u}})dv \; = \; \int_{\mathcal{B}} \mathbf{b}\, dv \; + \; \int_{\partial\mathcal{B}} \mathbf{t}\, da \tag{3.35}$$

or in terms of

$$\int_{\mathcal{B}} [\mathbf{b} \; - \; \frac{D}{Dt}(\rho\dot{\mathbf{u}})]dv \; + \; \int_{\partial\mathcal{B}} \mathbf{t}\, da \; = \; 0 \tag{3.36}$$

With the help of the *Divergence Theorem* the surface integral converts into a volume integral,

$$\boxed{\int_{\partial\mathcal{B}} \sigma_{ji}\, n_j \, da \; = \; \int_{\mathcal{B}} \sigma_{ji,j}\, dv} \tag{3.37}$$

Introducing $t_i \; = \; \sigma_{ji}\, n_j$ in the balance of linear momentum we find the differential equations of motion:

$$\boxed{\sigma_{ji,j} \; + \; b_i \; = \; \tfrac{D}{Dt}(\rho\dot{u}_i)} \quad \text{in} \quad dv \tag{3.38}$$

This differential equilibrium holds at every point in the interior of the body.

In statics, omitting inertia effects, the balance equations reduce to the differential equations of equilibrium, which read in index and in direct notations:

$$\sigma_{ji,j} \; + \; b_i \; = \; 0_i \quad \text{or} \quad div\boldsymbol{\sigma} + \mathbf{b} = \; \mathbf{0} \tag{3.39}$$

## 3.4 Balance of Angular Momentum

Angular momentum involves

$$\mathbf{h}_0 \;=\; \int_{\mathcal{B}_\mathcal{I}} (\mathbf{x} \times \rho \dot{\mathbf{u}})\, dv \tag{3.40}$$

where the pole is assumed to coincide with the origin $\mathbf{x}_0 \;=\; 0$. The moment of the distributed forces is

$$\mathbf{m}_0 \;=\; \int_{\mathcal{B}} (\mathbf{x} \times \mathbf{b}) dv \;+\; \int_{\partial\mathcal{B}} (\mathbf{x} \times \mathbf{t})\, da \tag{3.41}$$

The balance of angular momentum states

$$\boxed{\tfrac{D\mathbf{h}_0}{Dt} = \mathbf{m}_0} \tag{3.42}$$

$$\frac{D}{Dt}\int_{\mathcal{B}} (\mathbf{x} \times \rho \dot{\mathbf{u}}) dv \;=\; \int_{\mathcal{B}} (\mathbf{x} \times \mathbf{b}) dv \;+\; \int_{\partial\mathcal{B}} (\mathbf{x} \times \mathbf{t})\, da \tag{3.43}$$

The divergence theorem yields for the last term above

$$\int_{\mathcal{B}} [\mathbf{x} \times (\mathbf{b} + div\, \boldsymbol{\sigma}^t - \rho \ddot{\mathbf{u}})] dv + 2 \int_{\mathcal{B}} (\mathbf{1} \times \boldsymbol{\sigma}^t) dv \;=\; 0 \tag{3.44}$$

Application of the theorem of Cauchy : $\mathbf{b} + div\, \boldsymbol{\sigma}^t - \rho\, \frac{D}{Dt}\dot{\mathbf{u}} \;=\; 0$, renders

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^t} \tag{3.45}$$

This states that the stress tensor is symmetric, i.e. $\boldsymbol{\sigma}^t = \boldsymbol{\sigma}$, i.e. the *"Boltzmann Postulate"* of symmetric stress.

In index notation :

$$e_{ijk}\sigma_{jk} = 0 \rightarrow \sigma_{jk} = \sigma_{kj} \tag{3.46}$$

this infers,

$$\sigma_{23} - \sigma_{32} = 0;\; \sigma_{31} - \sigma_{13} = 0;\; \sigma_{12} - \sigma_{21} = 0 \tag{3.47}$$

Figure 3.6 illustrates the argument of moment equilibrium leading to the concept of conjugate shear stress and to the synmmetry of the stress tensor when $\boldsymbol{\sigma} \;=\; \boldsymbol{\sigma}^t$.

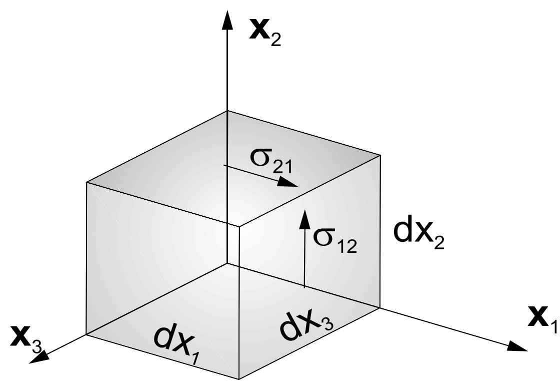


Figure 3.7: Equations of moment equilibrium: $\mathbf{m}_3 = 0 \rightarrow \sigma_{21} = \sigma_{12}$

# Chapter 4

# Elastic Material Models

Linear elasticity is the main staple of material models in solids and structures. The statement *'ut tensio sic vis'* attributed to ROBERT HOOKE (1635-1703) characterizes the behavior of a linear spring in which the deformations increase proportionally with the applied forces according to the anagram *'ceiiinossstuv'*. The original format of Hooke's law included the geometric properties of the wire test specimens, and therefore the spring constant did exhibit a pronounced size effect. The definition of the modulus of elasticity $E$, where

$$\sigma = E\,\epsilon \tag{4.1}$$

is attributed to THOMAS YOUNG (1773-1829). He expressed the proportional material behavior through the notion of a normalized force density and a normalized deformation measure, though the original formulation also did not entirely eliminate the size effect.
The tensorial character of stress was established by Cauchy, who defined the triaxial state of stress by three traction vectors using the celebrated tetraeder argument of equilibrium. The state of stress is described in terms of Cartesian coordinates by the second order tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{4.2}$$

The conjugate state of strain is a second order tensor with Cartesian coordinates,

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \tag{4.3}$$

which is normally expressed in terms of the symmetric part of the displacement gradient, if we restrict our attention to infinitesimal deformations. In the case of non-polar media we may confine our attention to stress measures, which are symmetric according to the axiom of L. Boltzmann,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^t \quad \text{or} \quad \sigma_{ij} = \sigma_{ji} \tag{4.4}$$

and the conjugate strain measures

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^t \quad \text{or} \quad \epsilon_{ij} = \epsilon_{ji} \tag{4.5}$$

where $i = 1, 2, 3$ and $j = 1, 2, 3$. As a result, the eigenvalues are real-valued and constitute the set of principal stresses and strains with zero shear components in the principal eigen-directions of the second order tensor. In contrast, non-symmetric stress and strain measures may exhibit complex conjugate principal values and maximuum normal stress and strain components in directions with non-zero shear components characteristic for micropolar Cosserat continua.
Restricting this exposition to symmetric stress and strain tensors they may be cast into vector form using the Voigt notation of crystal physics.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \tau_{12} & \tau_{23} & \tau_{31} \end{bmatrix}^t \tag{4.6}$$

and

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{22} & \epsilon_{33} & \gamma_{12} & \gamma_{23} & \gamma_{31} \end{bmatrix}^t \tag{4.7}$$

where $\tau_{ij} = \sigma_{ij}$, $\gamma_{ij} = 2\epsilon_{ij}, \forall i \neq j$. The vector form of stress and strain will allow us to formulate material models in matrix notation used predominantly in engineering.

## 4.1 Linear Elastic Material Behavior:

Generalization of the scalar format of Hooke's law is based on the notion that the triaxial state of stress is proportional to the triaxial state of strain through the linear transformation,

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{E}} : \boldsymbol{\epsilon} \quad \text{or} \quad \sigma_{ij} = \mathcal{E}_{ijkl}\epsilon_{kl} \tag{4.8}$$

Considering the symmetry of the stress and strain, the elasticity tensor involves in general 36 elastic moduli. This may be further reduced to 21 elastic constants, if we invoke major symmetry of the elasticity tensor, i.e.

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^t \quad \text{or} \quad \mathcal{E}_{ijkl} = \mathcal{E}_{klij} \quad \text{with} \quad \mathcal{E}_{ijkl} = \mathcal{E}_{ijlk} \quad \text{and} \quad \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} \tag{4.9}$$

The task of identifying 21 elastic moduli is simplified if we consider specific classes of symmetry, whereby orthotropic elasticity involves nine, and transversely anisotropic elasticity five elastic moduli.

### 4.1.1 Isotropic Linear Elasticity

In the case of isotropy the fourth order elasticity tensor has the most general representation,

$$\boldsymbol{\mathcal{E}} = a_o \mathbf{1} \otimes \mathbf{1} + a_1 \mathbf{1} \bar{\otimes} \mathbf{1} + a_2 \mathbf{1} \underline{\otimes} \mathbf{1} \quad or \quad \mathcal{E}_{ijkl} = a_o \delta_{ij}\delta_{kl} + a_1 \delta_{ik}\delta_{jl} + a_2 \delta_{il}\delta_{jk} \tag{4.10}$$

where $\mathbf{1} = [\delta_{ij}]$ stands for the second order unit tensor. The three parameter expression may be recast in terms of symmetric and skew symmetric fourth order tensor components as

$$\boldsymbol{\mathcal{E}} = a_o \mathbf{1} \otimes \mathbf{1} + b_1 \boldsymbol{\mathcal{I}} + b_2 \boldsymbol{\mathcal{I}}^{skew} \tag{4.11}$$

where the symmetric fourth order unit tensor reads

$$\boldsymbol{\mathcal{I}} = \frac{1}{2}[\mathbf{1} \bar{\otimes} \mathbf{1} + \mathbf{1} \underline{\otimes} \mathbf{1}] \quad or \quad \mathcal{I}_{ijkl} = \frac{1}{2}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] \tag{4.12}$$

and the skewed symmetric one

$$\boldsymbol{\mathcal{I}}^{skew} = \frac{1}{2}[\mathbf{1}\bar{\otimes}\mathbf{1} - \mathbf{1}\underline{\otimes}\mathbf{1}] \quad or \quad \mathcal{I}^{skew}_{ijkl} = \frac{1}{2}[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}] \tag{4.13}$$

Because of the symmetry of stress and strain the skewed symmetric contribution is inactive, $b_2 = 0$, thus isotropic linear elasticity the material behavior is fully described by two independent elastic constants. In short, the fourth order material stiffness tensor reduces to

$$\boldsymbol{\mathcal{E}} = \Lambda \mathbf{1} \otimes \mathbf{1} + 2G\boldsymbol{\mathcal{I}} \quad or \quad \mathcal{E}_{ijkl} = \Lambda \delta_{ij}\delta_{kl} + G[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] \tag{4.14}$$

where the two elastic constants $\Lambda, G$ are named after GABRIEL LAMÉ (1795-1870).

$$\Lambda = \frac{E\,\nu}{[1+\nu][1-2\nu]} \tag{4.15}$$

denotes the cross modulus, and

$$G = \frac{E}{2[1+\nu]} \tag{4.16}$$

designates the shear modulus which have a one-to-one relationship with the modulus of elasticity and Poisson's ratio, $E, \nu$.
In the absence of initial stresses and initial strains due to environmental effects, the linear elastic relation reduces to

$$\boldsymbol{\sigma} = \Lambda[\mathrm{tr}\boldsymbol{\epsilon}]\mathbf{1} + 2G\boldsymbol{\epsilon} \quad \text{or} \quad \sigma_{ij} = \Lambda \epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij} \tag{4.17}$$

Here the trace operation is the sum of the diagonal entries of the second order tensor corresponding to double contraction with the identity tensor $tr\boldsymbol{\epsilon} = \epsilon_{kk} = \mathbf{1} : \boldsymbol{\epsilon}$.
The inverse strain-stress relation $\boldsymbol{\epsilon} = \boldsymbol{\mathcal{C}} : \boldsymbol{\sigma}$ reads as follows for isotropic conditions:

$$\boldsymbol{\epsilon} = \frac{1}{\Lambda}[\mathrm{tr}\boldsymbol{\sigma}]\mathbf{1} + \frac{1}{2G}\boldsymbol{\epsilon} \quad \text{or} \quad \epsilon_{ij} = \frac{1}{\Lambda}\sigma_{kk}\delta_{ij} + \frac{1}{2G}\sigma_{ij} \tag{4.18}$$

where the fourth order compliance tensor writes in direct and index notations:

$$\boldsymbol{\mathcal{C}} = -\frac{\nu}{E}\mathbf{1} \otimes \mathbf{1} + \frac{1}{2G}\boldsymbol{\mathcal{I}} \quad or \quad C_{ijkl} = -\frac{\nu}{E}\delta_{ij}\delta_{kl} + \frac{1+\nu}{2E}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] \tag{4.19}$$

### 4.1.2 Matrix Format of Elastic Stiffness: $\boldsymbol{\sigma} = \boldsymbol{E}\,\boldsymbol{\epsilon}$

Isotropic linear elastic behavior may be cast in matrix format, using the Voigt notation of symmetric stress and strain tensors and the engineering definition of shear strain $\gamma_{ij} = 2\epsilon_{ij}$. The elastic stiffness matrix may be written for isotropic behavior as,

$$\boldsymbol{E} = \left[\begin{array}{ccc|ccc} \Lambda+2G & \Lambda & \Lambda & & & \\ \Lambda & \Lambda+2G & \Lambda & & 0 & \\ \Lambda & \Lambda & \Lambda+2G & & & \\ \hline & & & G & & \\ & 0 & & & G & \\ & & & & & G \end{array}\right] \tag{4.20}$$

### 4.1.3 Matrix Format of Elastic Compliance: $\boldsymbol{\epsilon} = \boldsymbol{C}\,\boldsymbol{\sigma}$

In the isotropic case the normal stress $\sigma_{11}$ gives rise to three normal strain contributions, the direct strain $\epsilon_{11} = \frac{1}{E}\sigma_{11}$ and the normal strains $\epsilon_{22} = -\frac{\nu}{E}\sigma_{11}, \epsilon_{33} = \frac{\nu}{E}\sigma_{11}$ because of the cross effect attributed to SIMÉON DENIS POISSON (1781-1840). Using the principle of superposition, the additional strain contributions due to $\sigma_{22}$ and $\sigma_{33}$ enter the compliance relation for isotropic elasticity in matrix format,

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} \tag{4.21}$$

In the isotropic case the shear response is entirely decoupled from the direct response of the normal components. Thus the compliance matrix expands into the partitioned form

$$\boldsymbol{C} = \frac{1}{E} \left[ \begin{array}{ccc|ccc} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & 0 & \\ -\nu & -\nu & 1 & & & \\ \hline & & & 2[1+\nu] & & \\ & 0 & & & 2[1+\nu] & \\ & & & & & 2[1+\nu] \end{array} \right] \tag{4.22}$$

Note, isotropy entirely decouples the shear response from the normal stress-strain response. This cross effect of POISSON is illustrated in Figure 2, which shows the interaction of lateral and axial deformations under axial compression. It is intriguing that in his original work a value of $\nu = 0.25$ was proposed by S. POISSON based on molecular considerations.

### 4.1.4 Canonical Format of Isotropic Elasticity:

Decomposing the stress and strain tensors into spherical and deviatoric components

$$\boldsymbol{s} = \boldsymbol{\sigma} - \sigma_{vol}\mathbf{1} \quad \text{where} \quad \sigma_{vol} = \frac{1}{3}[tr\boldsymbol{\sigma}] \tag{4.23}$$

$$\boldsymbol{e} = \boldsymbol{\epsilon} - \epsilon_{vol}\mathbf{1} \quad \text{where} \quad \epsilon_{vol} = \frac{1}{3}[tr\boldsymbol{\epsilon}] \tag{4.24}$$

leads to the stress deviator

$$\boldsymbol{s} = \begin{bmatrix} \frac{1}{3}[2\sigma_{11} - \sigma_{22} - \sigma_{33}] & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \frac{1}{3}[2\sigma_{22} - \sigma_{33} - \sigma_{11}] & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \frac{1}{3}[2\sigma_{33} - \sigma_{11} - \sigma_{22}] \end{bmatrix} \tag{4.25}$$

and the strain deviator

$$\boldsymbol{e} = \begin{bmatrix} \frac{1}{3}[2\epsilon_{11} - \epsilon_{22} - \epsilon_{33}] & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \frac{1}{3}[2\epsilon_{22} - \epsilon_{33} - \epsilon_{11}] & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \frac{1}{3}[2\epsilon_{33} - \epsilon_{11} - \epsilon_{22}] \end{bmatrix} \tag{4.26}$$

which have the property $tr\boldsymbol{s} = 0$ and $tr\boldsymbol{e} = 0$. The decomposition decouples the volumetric from the distortional response, because of the underlying orthogonality of the spherical and deviatoric partitions, $\boldsymbol{s} : [\sigma_{vol}\mathbf{1}] = 0$ and $\boldsymbol{e} : [\epsilon_{vol}\mathbf{1}] = 0$. The decoupled response reduces the elasticity tensor to the scalar form,

$$\sigma_{vol} = 3\,K\epsilon_{vol} \quad \text{and} \quad \boldsymbol{s} = 2\,G\boldsymbol{e} \tag{4.27}$$

The bulk and the shear moduli,

$$K = \frac{E}{3[1-2\nu]} = \Lambda + \frac{2}{3}G \quad \text{and} \quad G = \frac{E}{2[1+\nu]} = \frac{3}{2}[K - \Lambda] \tag{4.28}$$

define the volumetric and the deviatoric material stiffness.
Consequently, the internal strain energy density expands into the canonical form of two decoupled contributions

$$2W = \boldsymbol{\sigma} : \boldsymbol{\epsilon} = [\sigma_{vol}\mathbf{1}] : [\epsilon_{vol}\mathbf{1}] + \boldsymbol{s} : \boldsymbol{e} = 9\,K\epsilon_{vol}^2 + 2G\boldsymbol{e} : \boldsymbol{e} \tag{4.29}$$

such that the positive strain energy argument delimits the range of possible values of Poisson's ratio to $-1 \leq \nu \leq 0.5$

### 4.1.5 Temperature Effects in Linear Isotropic Elasticity

In the case of isotropic material behavior, with no directional properties, the size of a representative volume element may change due to thermal effects or shrinkage and swelling, but it will not distort. Consequently, the expansion is purely volumetric, i.e. identical in all directions. Using direct and index notation, the additive decomposition of strain into elastic and initial volumetric components, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_e + \boldsymbol{\epsilon}_o$, leads to the following extension of the elastic compliance relation:

$$\boldsymbol{\epsilon} = -\frac{\nu}{E}[tr\boldsymbol{\sigma}]\mathbf{1} + \frac{1}{2G}\boldsymbol{\sigma} + \epsilon_o\mathbf{1} \quad \text{or} \quad \epsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1}{2G}\sigma_{ij} + \epsilon_o\delta_{ij} \tag{4.30}$$

where $\boldsymbol{\epsilon}_o = \alpha\Delta T\mathbf{1}$ denotes the initial volumetric strain e.g. due to thermal expansion when the temperature changes from the reference temperature, $\Delta T = T - T_o$. The inverse relation reads

$$\boldsymbol{\sigma} = \Lambda[tr\boldsymbol{\epsilon}]\mathbf{1} + 2G\boldsymbol{\epsilon} - 3\epsilon_o K\mathbf{1} \quad \text{or} \quad \sigma_{ij} = \Lambda\epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij} - 3\epsilon_o K\delta_{ij} \tag{4.31}$$

Rewriting this equation in matrix notation for $K$ and $G$ we have:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\end{bmatrix} = \begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G \\ K-\frac{2}{3}G & K+\frac{4}{3}G & K-\frac{2}{3}G \\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{4}{3}G \end{bmatrix} \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\end{bmatrix} - 3K\alpha\Delta T \begin{bmatrix}1\\1\\1\end{bmatrix} \tag{4.32}$$

Considering the special case of plane stress, $\sigma_{33} = 0$, the stress-strain relations reduce in the presence of initial volumetric strains to:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu\\ \nu & 1\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\end{bmatrix} - \frac{E}{1-\nu}\alpha\Delta T\begin{bmatrix}1\\1\end{bmatrix} \tag{4.33}$$

where the shear components are not affected by the temperature change in the case of isotropy.

### Free Thermal Expansion

Under stress free conditions the thermal expansion $\boldsymbol{\epsilon}_o = \alpha[T - T_o]\mathbf{1}$ leads to $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_o$, i.e.

$$\epsilon_{11} = \alpha[T - T_o] \quad (4.34)$$

$$\epsilon_{22} = \alpha[T - T_o] \quad (4.35)$$

$$\epsilon_{33} = \alpha[T - T_o] \quad (4.36)$$

Thus the change of temperature results in free thermal expansion, while the mechanical stress remains zero under zero confinement, $\boldsymbol{\sigma} = \boldsymbol{\mathcal{E}} : \boldsymbol{\epsilon}_e = \mathbf{0}$.

### Thermal Stress under Full Restraint

In contrast to the unconfined situation above, the thermal expansion is equal and opposite to the elastic strain $\boldsymbol{\epsilon}_e = -\boldsymbol{\epsilon}_o$ under full confinement, when $\boldsymbol{\epsilon} = 0$. In the case of plane stress, the temperature change $\Delta T = T - T_o$ leads to the thermal stresses

$$\sigma_{11} = -\frac{E}{1-\nu}\alpha[T - T_o] \quad (4.37)$$

$$\sigma_{22} = -\frac{E}{1-\nu}\alpha[T - T_o] \quad (4.38)$$