CVEN 7511

Computational Mechanics of Solids and Structures

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Original Version of Class Notes

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Chapter 1

Fundamentals of Continuum Mechanics

Abstract

In this section, two topics of Continuum Mechanics will be reviewed:

- Kinematics of Motion : $\mathbf{X}, \mathbf{x}, \mathbf{u}$ (Strain $\boldsymbol{\epsilon} \rightarrow \dot{\boldsymbol{\epsilon}}$).
- Balance Laws : conservation of mass, linear and angular momenta. (Stress $\boldsymbol{\sigma} \rightarrow \dot{\boldsymbol{\sigma}}$).

1.1 Kinematics of Motion:

The description of motion can be divided into :

- Lagrange Coordinates : (Material Description)

$$\mathbf{X} = X_A \mathbf{e}_A \quad \text{where} \quad A = 1, 2, 3 \tag{1.1}$$

- Euler Coordinates : (Spatial Description)

$$\mathbf{x} = x_i \mathbf{e}_i \quad \text{where} \quad i = 1, 2, 3 \tag{1.2}$$

The scalar temperature field may be represented by :

- Lagrangian Coordinates : $T = T(\mathbf{X}, t)$ - material description.
- Eulerian Coordinates : $T = T(\mathbf{x}, t)$ - spatial description.

Jacobian as Deformation Gradient :

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \Rightarrow d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} \tag{1.3}$$

Then mapping of the material line element $d\mathbf{X}$ from the reference into current configuration is defined by :

$$\begin{bmatrix} dx_i \\ dx_j \\ dx_k \end{bmatrix} = \begin{bmatrix} \frac{dx_i}{dX_A} & \frac{dx_i}{dX_B} & \frac{dx_i}{dX_C} \\ \frac{dx_j}{dX_A} & \frac{dx_j}{dX_B} & \frac{dx_j}{dX_C} \\ \frac{dx_k}{dX_A} & \frac{dx_k}{dX_B} & \frac{dx_k}{dX_C} \end{bmatrix} \begin{bmatrix} dX_A \\ dX_B \\ dX_C \end{bmatrix} \tag{1.4}$$

Restriction : for unique one-to-one mapping $\boxed{\det \mathbf{F} \neq 0}$ where $J = \det \mathbf{F}$.

1.2 Polar Decomposition :

Polar decomposition theorem states that the deformation gradient tensor $\mathbf{F}$ may be decomposed uniquely into a positive definite tensor and a proper orthogonal tensor, i.e. the *right* $\mathbf{U}$ or *left* $\mathbf{V}$ *stretch* tensor plus the *rotation* $\mathbf{R}$ tensor.

1. Right Description : $\boxed{\mathbf{F} = \mathbf{R} \cdot \mathbf{U}}$
   where: $\det \mathbf{R} = 1, \qquad \mathbf{R}^t \cdot \mathbf{R} = \mathbf{1}, \qquad \mathbf{R} \cdot \mathbf{R}^t = \mathbf{1},$
   thus $\boxed{\mathbf{U} = \mathbf{U}^t}$ such that $\lambda_U > 0$ and

$$\mathbf{U}^2 = \mathbf{U}^t \cdot \mathbf{R}^t \cdot \mathbf{R} \cdot \mathbf{U} = \mathbf{F}^t \cdot \mathbf{F} \tag{1.5}$$

2. Left Description : $\boxed{\mathbf{F} = \mathbf{V} \cdot \mathbf{R}}$ with

$$\mathbf{V}^2 = \mathbf{F} \cdot \mathbf{F}^t = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{R}^t \cdot \mathbf{V} \tag{1.6}$$

   and $\lambda_V = \lambda_U$

Physical Meaning of $\mathbf{U}$ and $\mathbf{V}$:

- Right Stretch : $d\mathbf{x} = \mathbf{R} \cdot \mathbf{U} \cdot d\mathbf{X}$, where $d\mathbf{X}$ is first stretched and then rotated.
- Left Stretch : $d\mathbf{x} = \mathbf{V} \cdot \mathbf{R} \cdot d\mathbf{X}$, where $d\mathbf{X}$ is first rotated and then stretched.

If $\mathbf{F}$ is nonsingular $\Rightarrow \det \mathbf{F} \neq 0$, then there is a *unique* decomposition into a proper orthogonal tensor $\mathbf{R}$ and a *positive definite* tensor $\mathbf{U}$ or $\mathbf{V}$.

Logarithmic Hencky Strain [1928]: in the uniaxial case the logarithmic strain is defined by integrating the stretch rate:

$$\epsilon_{ln} = \int_{L_0}^{L} \frac{d\ell}{\ell} = ln\frac{L}{L_0} = ln\lambda \quad \text{where} \quad \lambda = \frac{L}{L_0}. \tag{1.7}$$

Triaxial extension of logarithmic strain:

- Lagrangian format : $\boldsymbol{\epsilon}_{ln}^{U} = ln\mathbf{U}$, with principal coordinates which are defined by $\mathbf{e}_U$)
- Eulerian format : $\boldsymbol{\epsilon}_{ln}^{V} = ln\mathbf{V}$, with principal Coordinates which are defined by $\mathbf{e}_V = \mathbf{R} \cdot \mathbf{e_U}$.

Spectral Representation :

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \, \mathbf{e}_U^i \otimes \mathbf{e}_U^i \quad \text{and} \quad \boldsymbol{\epsilon}_{ln}^{U} = \sum_{i=1}^{3} (ln\lambda_i) \, \mathbf{e}_U^i \otimes \mathbf{e}_U^i \tag{1.8}$$

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \, \mathbf{e}_V^i \otimes \mathbf{e}_V^i \quad \text{and} \quad \boldsymbol{\epsilon}_{ln}^{V} = \sum_{i=1}^{3} (ln\lambda_i) \, \mathbf{e}_V^i \otimes \mathbf{e}_V^i \tag{1.9}$$

1.3 Lagrangian Strain Measure :

Extensional deformation

$$ds^2 - dS^2 = d\mathbf{x}^t \cdot d\mathbf{x} - d\mathbf{X}^t \cdot d\mathbf{X} = d\mathbf{X}^t \cdot (\mathbf{F}^t \cdot \mathbf{F} - \mathbf{1}) \cdot d\mathbf{X} \tag{1.10}$$

The Green-Lagrange strain is related to the right stretch tensor by

$$\boxed{\mathbf{E}_G = \tfrac{1}{2} \left(\mathbf{F}^t \cdot \mathbf{F} - \mathbf{1} \right) = \tfrac{1}{2} \left(\mathbf{U}^2 - \mathbf{1} \right)} \tag{1.11}$$

In indicial form,

$$\mathbf{F} = \frac{\partial x_i}{\partial X_A} \, \mathbf{e}_i \otimes \mathbf{e}_A \tag{1.12}$$

$$\mathbf{U}^2 = \frac{\partial x_i}{\partial X_A} \cdot \frac{\partial x_i}{\partial X_B} \tag{1.13}$$

$$E_{AB}^{G} = \frac{1}{2} \left(\frac{\partial x_i}{\partial X_A} \cdot \frac{\partial x_i}{\partial X_B} - \delta_{AB} \right) \tag{1.14}$$

Generalized Lagrangian Strain [Doyle-Erickson 1956]:

for m=0

$$\boxed{\mathbf{E}_0 = ln\mathbf{U}} \tag{1.15}$$

and for m=1,2,...

$$\boxed{\mathbf{E}_m = \tfrac{1}{m} \left(\mathbf{U}^m - \mathbf{1} \right)} \tag{1.16}$$

1-dim examples with $\lambda = \frac{L}{L_0}$:

$$\begin{aligned} E_0 &= ln\lambda \\ E_1 &= \lambda - 1 \\ E_2 &= \tfrac{1}{2} \left(\lambda^2 - 1 \right) \end{aligned} \tag{1.17}$$

Strain-Displacement Relationship

The displacement description of motion defines the new location of the material particle $\mathbf{X}$ in terms of :

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) = \mathbf{u}(\mathbf{X}, t) + \mathbf{X} \tag{1.18}$$

The deformation gradient is given as

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{1} = \mathbf{H} + \mathbf{1} \quad \text{where} \quad \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{H} \tag{1.19}$$

With $\mathbf{F} = \mathbf{H} + \mathbf{1}$, we may develop the Lagrangian strain-displacement relationship

$$\mathbf{U}^2 = \mathbf{F}^t \cdot \mathbf{F} = \mathbf{1} + \mathbf{H} + \mathbf{H}^t + \mathbf{H}^t \cdot \mathbf{H} \tag{1.20}$$

1.4 Eulerian Strain Measure :

Spatial description of extensional deformation

$$ds^2 - dS^2 = d\mathbf{x}^t \cdot (\mathbf{1} - \mathbf{F}^{-t} \cdot \mathbf{F}^{-1}) \cdot d\mathbf{x} \tag{1.21}$$

The Almansi strain tensor is in terms of the left stretch tensor

$$\boxed{\mathbf{e}_A = \tfrac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-t} \cdot \mathbf{F}^{-1}\right) = \tfrac{1}{2}\left(\mathbf{1} - (\mathbf{V}^2)^{-1}\right)} \tag{1.22}$$

Generalized Eulerian Strain [Doyle-Erickson 1956]:

for m=0

$$\boxed{\mathbf{e}_0 = ln\mathbf{V}} \tag{1.23}$$

and for m=-1,-2,...

$$\boxed{\mathbf{e}_m = \tfrac{1}{m}\left(\mathbf{V}^m - \mathbf{1}\right)} \tag{1.24}$$

1-dim examples when $\frac{1}{\lambda} = \frac{L_0}{L}$:

$$\begin{aligned} e_0 &= -ln\tfrac{1}{\lambda} \\ e_{-1} &= 1 - \tfrac{1}{\lambda} \\ e_{-2} &= \tfrac{1}{2}\left(1 - \tfrac{1}{\lambda^2}\right) \end{aligned} \tag{1.25}$$

Strain-Displacement Relationship

In the spatial description the motion is described by:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t) \tag{1.26}$$

The spatial displacement gradient is defined as:

$$\mathbf{h} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{1} - \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \mathbf{1} - \mathbf{F}^{-1} \tag{1.27}$$

or

$$\mathbf{F}^{-1} = \mathbf{1} - \mathbf{h} \tag{1.28}$$

$$\mathbf{V}^{-2} = \mathbf{F}^{-t} \cdot \mathbf{F}^{-1} = \mathbf{1} - \mathbf{h} - \mathbf{h}^t + \mathbf{h}^t \cdot \mathbf{h} \tag{1.29}$$

Substituting into the definition of the Almansi strain $\mathbf{e}_A$ leads to the Almansi strain-displacement relationship

$$\boxed{\mathbf{e}_A = \tfrac{1}{2}(\mathbf{h} + \mathbf{h}^t - \mathbf{h}^t \cdot \mathbf{h})} \tag{1.30}$$

1.5 Infinitesimal Deformations:

If $\det \mathbf{H} << 1 \Rightarrow \det(\mathbf{H}^t \cdot \mathbf{H}) \simeq 0$, then

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^t\right) \simeq \frac{1}{2}\left(\mathbf{h} + \mathbf{h}^t\right) \tag{1.31}$$

For *"Infinitesimal Deformations"* the spatial and the material displacement gradients coincide,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} \sim \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \tag{1.32}$$

Additive decomposition into symmetric and skew-symmetric components leads to

$$\mathbf{H} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^t) + \frac{1}{2}(\mathbf{H} - \mathbf{H}^t) \tag{1.33}$$

where the symmetric component defines the traditional linearized strain tensor

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}) \tag{1.34}$$

and where the skew-symmetric component defines the infinitesimal rotation tensor

$$\omega_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}) \tag{1.35}$$

such that $\epsilon_{ij} = \epsilon_{ji}$ and $\omega_{ij} = -\omega_{ji}$.

1.6 The Velocity Gradient:

Considering the Eulerian format of the velocity field:

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) \tag{1.36}$$

the *Velocity Gradient* is defined as :

$$\boldsymbol{\ell} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \tag{1.37}$$

Additive decomposition into symmetric and skew-symmetric components leads to

$$\boldsymbol{\ell} = \mathbf{d} + \mathbf{w} = \frac{1}{2}(\boldsymbol{\ell} + \boldsymbol{\ell}^t) + \frac{1}{2}(\boldsymbol{\ell} - \boldsymbol{\ell}^t) \tag{1.38}$$

The rate of deformation tensor is the symmetric part

$$\mathbf{d} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) \tag{1.39}$$

The spin tensor is the skew-symmetric part

$$\mathbf{w} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}) \tag{1.40}$$

1.7 The Rate of Deformation Tensor:

Given

$$ds^2 = d\mathbf{x}^t \cdot d\mathbf{x} = d\mathbf{X}^t \cdot (\mathbf{F}^t \cdot \mathbf{F}) \cdot d\mathbf{X} \tag{1.41}$$

then the material time derivative is

$$\frac{D}{Dt}(ds^2) = d\mathbf{X}^t \cdot (\mathbf{F}^t \cdot \dot{\mathbf{F}} + \dot{\mathbf{F}}^t \cdot \mathbf{F}) d\mathbf{X} \tag{1.42}$$

Given $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ the material time derivative yields $\dot{\mathbf{F}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$.
Relationship between rate of deformation gradient and velocity gradient:

$$\boxed{\dot{\mathbf{F}} = \boldsymbol{\ell} \cdot \mathbf{F}} \tag{1.43}$$

The difference between the spatial description of the temperature field $T = T(\mathbf{x}, t)$ and the material description $T = T(\mathbf{X}, t)$ leads to the important difference of derivatives as follows:

- Spatial time derivative : $\dot{T} = \dot{T}(\mathbf{x}, t)$
- Material time derivative : $\frac{D}{Dt}(T) = \dot{T} + \frac{\partial T}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}}$

Considering the *"material time derivative"* of the line element ds we have

$$\frac{D}{Dt}(ds^2) = d\mathbf{X}^t \cdot (\mathbf{F}^t \cdot \boldsymbol{\ell} \cdot \mathbf{F} + \mathbf{F}^t \cdot \boldsymbol{\ell}^t \cdot \mathbf{F}) \cdot d\mathbf{X} \tag{1.44}$$

and in spatial form

$$\frac{D}{Dt}(ds^2) = d\mathbf{x}^t \cdot (\boldsymbol{\ell} + \boldsymbol{\ell}^t) \cdot d\mathbf{x} = 2d\mathbf{x}^t \cdot \mathbf{d} \cdot d\mathbf{x} \tag{1.45}$$

Hence the rate of deformation tensor measures the rate of extensional deformation of ds

$$\frac{1}{2}\frac{D}{Dt}(ds^2) = d\mathbf{x}^t \cdot \mathbf{d} \cdot d\mathbf{x} \tag{1.46}$$

1.8 Reynold's Transport Theorem :

The material time derivative of a physical quantity in the spatial description involves two terms

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} \mid_X = \frac{\partial}{\partial t} \mid_x + \mathbf{v} \cdot grad} \tag{1.47}$$

For the scalar temperature field $T = T(\mathbf{x}, t)$ we have

$$\frac{DT}{Dt} = \frac{\partial T(\mathbf{X}, t)}{\partial t} = \frac{\partial T(\mathbf{x}, t)}{\partial t} + \frac{\partial T}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial t} = \dot{T} \mid_x + grad\, T \cdot \mathbf{v} \tag{1.48}$$

Recalling that $\dot{\mathbf{F}} = \boldsymbol{\ell} \cdot \mathbf{F}$ and $\dot{\mathbf{F}}^t = \mathbf{F}^t \cdot \boldsymbol{\ell}^t$, we consider the material time derivative of the right polar decomposition $\mathbf{U}^2 = \mathbf{F}^t \cdot \mathbf{F}$:

$$\frac{D}{Dt}(\mathbf{U}^2) = \frac{D}{Dt}(\mathbf{F}^t \cdot \mathbf{F}) = \dot{\mathbf{F}}^t \cdot \mathbf{F} + \mathbf{F}^t \cdot \mathbf{F} = \mathbf{F}^t \cdot \boldsymbol{\ell}^t \cdot \mathbf{F} + \mathbf{F}^t \cdot \boldsymbol{\ell} \cdot \mathbf{F} = \mathbf{F}^t \cdot (\boldsymbol{\ell}^t + \boldsymbol{\ell}) \cdot \mathbf{F} = 2\mathbf{F}^t \cdot \mathbf{d} \cdot \mathbf{F} \tag{1.49}$$

From the Green-Lagrange strain $\mathbf{E}_G = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1})$, we find the relationship with the rate of deformation tensor

$$\dot{\mathbf{E}}_G = \frac{1}{2}(\mathbf{U} \cdot \dot{\mathbf{U}} + \dot{\mathbf{U}} \cdot \mathbf{U}) = \mathbf{F}^t \cdot \mathbf{d} \cdot \mathbf{F} \tag{1.50}$$

1.9 Lagrangian Strain Rate:

From the right polar decomposition of $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$, the time rate of the deformation gradient is

$$\dot{\mathbf{F}} = \dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}} \tag{1.51}$$

With $\boxed{\dot{\mathbf{F}} = \boldsymbol{\ell} \cdot \mathbf{F}}$, the left hand side expands into

$$\boldsymbol{\ell} \cdot \mathbf{R} \cdot \mathbf{U} = \dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}} \tag{1.52}$$

After multiplication with $\mathbf{U}^{-1}$ and $\mathbf{R}^t$ we get

$$\boldsymbol{\ell} = \dot{\mathbf{R}} \cdot \mathbf{R}^t + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^t \tag{1.53}$$

where $\boxed{\dot{\mathbf{R}} \cdot \mathbf{R}^t = \boldsymbol{\Omega} = -\boldsymbol{\Omega}^t}$ denotes the Rate of the Material Rotation Tensor.

The velocity gradient decomposes into symmetric and skew-symmetric components

$$\mathbf{d} = \frac{1}{2}\mathbf{R} \cdot (\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}) \cdot \mathbf{R}^t \tag{1.54}$$

and

$$\mathbf{w} = \dot{\mathbf{R}} \cdot \mathbf{R}^t + \frac{1}{2}\mathbf{R} \cdot (\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}}) \cdot \mathbf{R}^t \tag{1.55}$$

Remark : The tensor $\boldsymbol{\Omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^t$ defines the rate of material rotation without stretching. Only when there is no change of stretch, the spin tensor $\mathbf{w}$ coincides with the rate of material rotation tensor $\mathbf{\Omega}$. In general, the rate of deformation tensor $\mathbf{d}$ does not coincide with the normalized rate of the right stretch tensor $\dot{\mathbf{U}}.\mathbf{U^{-1}}$ because of the superposed rotation $\mathbf{R} \neq \mathbf{0}$. This leads to the concept of the *" rotation-neutralized"* intermediate configuration which should be used to define rotation-free constitutive rate equations.

$$\mathbf{d}_{RN} = \mathbf{R}^t \cdot \mathbf{d} \cdot \mathbf{R} = \frac{1}{2}\,(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}) \tag{1.56}$$

1.10 Eulerian Strain Rate:

From the left polar decomposition of the Jacobian : $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$, the time rate of the Jacobian is

$$\dot{\mathbf{F}} = \dot{\mathbf{V}} \cdot \mathbf{R} + \mathbf{V} \cdot \dot{\mathbf{R}} \tag{1.57}$$

Using $\dot{\mathbf{F}} = \boldsymbol{\ell} \cdot \mathbf{F}$, then the left hand side expands into

$$\boldsymbol{\ell} \cdot \mathbf{V} \cdot \mathbf{R} = \dot{\mathbf{V}} \cdot \mathbf{R} + \mathbf{V} \cdot \dot{\mathbf{R}} \tag{1.58}$$

or after multiplication with $\mathbf{R}^t$ and $\mathbf{V}^{-1}$ we get

$$\boldsymbol{\ell} = \dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^t \cdot \mathbf{V}^{-1} \tag{1.59}$$

Then, the velocity gradient decomposes into symmetric and skew-symmetric components

$$\mathbf{d} = \frac{1}{2}\,(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \dot{\mathbf{V}}) \tag{1.60}$$

$$\mathbf{w} = \frac{1}{2}\,(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \dot{\mathbf{V}}) + \mathbf{V} \cdot \mathbf{\Omega} \cdot \mathbf{V}^{-1} \tag{1.61}$$

Remark : The rate of deformation tensor $\mathbf{d}$ coincides with the normalized rate of the left stretch tensor, while the spin tensor $\mathbf{w}$ does not agree with the rate of material rotation $\mathbf{\Omega}$ when stretching takes place with $\mathbf{V} \neq \mathbf{1}$ and $\dot{\mathbf{V}} \neq \mathbf{0}$

Chapter 2

Balance Laws:

The balance laws comprise statements as follows:

1. Balance of Linear Momentum.

2. Balance of Angular Momentum.

3. Balance of Mass.

4. Balance of Energy (First Law of Thermodynamics).

" Forces" are measured *indirectly* through their action on deformable solids.
Distributed forces include:

- $\mathbf{b}$: body force/unit volume(i. e. density).

- $\mathbf{t}_n$: surface traction.

Integrating the distributed forces over part I of the body $\mathcal{B}$ defines the resultant force vector

$$\mathbf{f}_I = \int_{\mathcal{B}_\mathcal{I}} \mathbf{b}\, dv + \int_{\partial\mathcal{B}_\mathcal{I}} \mathbf{t}_n\, da \tag{2.1}$$

2.1 Balance of Linear Momentum:

Linear momentum is defined as:

$$\mathbf{i} = \int_{\mathcal{B}_\mathcal{I}} \rho\, \dot{\mathbf{x}}\, dv \tag{2.2}$$

Application of Newton's second law $\sum \mathbf{f} = \mathbf{m} \cdot \mathbf{a}$ to the control volume of the body

$$\frac{D}{Dt}\mathbf{i} = \mathbf{f} \tag{2.3}$$

"Dynamic equilibrium" or the balance of linear momentum may be expressed as

$$\int_{\mathcal{B}_I} \frac{D}{Dt}(\rho\dot{\mathbf{x}})dv = \int_{\partial\mathcal{B}_\mathcal{I}} \mathbf{b}\, dv + \int_{\mathcal{B}_I} \mathbf{t}_n\, da \tag{2.4}$$

or in terms of

$$\int_{\mathcal{B}_I} (\mathbf{b} \;-\; \frac{D}{Dt}(\rho\dot{\mathbf{x}}))dv \;+\; \int_{\partial\mathcal{B}} \mathbf{t}_n \, da \;=\; 0 \tag{2.5}$$

Cauchy Lemma : states pointwise balance of surface tractions across any surface in the interior of the body.

$$\mathbf{t}_n(\mathbf{x}, \mathbf{n}) \;+\; \mathbf{t}_n(\mathbf{x}, -\mathbf{n}) \;=\; \mathbf{0} \tag{2.6}$$

or

$$\boxed{\mathbf{t}_n(\mathbf{x}, \mathbf{n}) \;=\; -\mathbf{t}_n(\mathbf{x}, -\mathbf{n})} \tag{2.7}$$

2.2 Cauchy's First Theorem:

The stress tensor is a linear mapping of the stress vector $\mathbf{t}_n$ onto the normal vector $\mathbf{n}$.

$$\mathbf{t}_n \;=\; \boldsymbol{\sigma}^t(\mathbf{x}) \cdot \mathbf{n} \tag{2.8}$$

In indicial notation,

$$t_i \;=\; \sigma_{ji} \, n_j \tag{2.9}$$

Considering elementary tetrahedron:

$$\mathbf{t}_n \;=\; \mathbf{n}_1 \, \mathbf{t}_1 \;+\; \mathbf{n}_2 \, \mathbf{t}_2 \;+\; \mathbf{n}_3 \, \mathbf{t}_3 \tag{2.10}$$

The stress tensor $\boldsymbol{\sigma}$ is simply composed of the coordinates of stress vectors on three mutually orthogonal planes

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.11}$$

In our notation of σ_{ij}, the first subscript i refers to normal direction of the *area* element and the subscript j refers to the direction of the *traction*.

Considering the equilibrium in x_1 direction : $\sum f_{x1} = 0$.

$$\mathbf{t}_1 da = \sigma_{11} n_1 da + \sigma_{21} n_2 da + \sigma_{31} n_3 da \tag{2.12}$$

then

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \tag{2.13}$$

With the help of the *Divergence Theorem* we find

$$\boxed{\int_{\partial\mathcal{B}} \sigma_{ji} \, n_j \, da \;=\; \int_{\mathcal{B}} \sigma_{ji,j} \, dv} \tag{2.14}$$

The equation of motion by Cauchy states the balance between the body forces and surface tractions when inertia forces remain negligible:

$$\int_{\mathcal{B}} b_i dv \;+\; \int_{\partial\mathcal{B}} t_i^n \, da \;=\; 0 \tag{2.15}$$

can be rewritten as

$$\boxed{\sigma_{ji,j} \;+\; b_i \;=\; 0} \tag{2.16}$$

"Cauchy's First Theorem" states pointwise equilibrium in the interior of the body.

In the dynamic case, the balance equations generalize to

$$\sigma_{ji,j} \;+\; b_i \;=\; \frac{D}{Dt}(\rho \dot{x}_i) \tag{2.17}$$

2.3 Balance of Mass:

The material time derivative of volume integral is comprised of two terms, one volume and one surface integral

$$\frac{D}{Dt}(\int_{\mathcal{B}} \Phi \rho dv) \;=\; \int_{\mathcal{B}} \frac{\partial}{\partial t}(\Phi\rho)\, dv \;+\; \int_{\partial\mathcal{B}} \Phi\, \rho\, \mathbf{v}\cdot\mathbf{n}\, da \tag{2.18}$$

Using the divergence theorem the above equation reduces to:

$$\frac{D}{Dt}(\int_{\mathcal{B}} \Phi\rho)dv = \int_{\mathcal{B}} \left[\Phi\, [\frac{\partial \rho}{\partial t} \;+\; div\,(\rho\mathbf{v})] \;+\; \rho[\frac{d\Phi}{dt} + \mathbf{v}\cdot grad\,\Phi] \right] dv \tag{2.19}$$

If we assume that the function $\Phi = 1$, then this reduces to an integral statement of mass conservation

$$\frac{D}{Dt}\int_{\mathcal{B}} \rho dv \;=\; \int_{\mathcal{B}} \left[\frac{\partial \rho}{\partial t} \;+\; div\,(\rho\,\mathbf{v})\right] dv \;=\; 0 \tag{2.20}$$

or pointwise it must hold

$$\boxed{\tfrac{\partial \rho}{\partial t} \;+\; div(\rho\mathbf{v}) \;=\; 0} \tag{2.21}$$

Expanding the divergence term

$$\frac{\partial \rho}{\partial t} \;+\; (grad\rho)\,\mathbf{v} \;+\; \rho\, div\mathbf{v} \;=\; 0 \tag{2.22}$$

Finally, the *Continuity Condition* at each point is simply

$$\boxed{\tfrac{D\rho}{Dt} \;+\; \rho\, div\mathbf{v} \;=\; 0} \tag{2.23}$$

In indicial notation

$$\frac{D\rho}{Dt} + \frac{\partial v_i}{\partial x_i} = 0 \tag{2.24}$$

where

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} \tag{2.25}$$

The continuity condition reduces the material time derivative of linear momentum to

$$\frac{D}{Dt}\int_{\mathcal{B}} \rho\dot{\mathbf{x}} = \int_{\mathcal{B}} \frac{\partial}{\partial t}(\rho\dot{\mathbf{x}})dv + \int_{\partial\mathcal{B}} \dot{\mathbf{x}}\,\rho\dot{\mathbf{x}}\cdot\mathbf{n}da = \int_{\mathcal{B}} \rho\frac{D\dot{\mathbf{x}}}{Dt}dv \tag{2.26}$$

Material Description of Mass Balance:

Considering $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$, with $\boxed{\det \mathbf{F} = J = \frac{dv}{dV}}$, then

$$\int_{\mathcal{B}} dv = \int_{\mathcal{B}_0} J dV \tag{2.27}$$

There is no mass flow, but only mapping of geometry.

For mass conservation,

$$\int_{\mathcal{B}} \rho \, dv = \int_{\mathcal{B}_0} \rho_0 \, dV \tag{2.28}$$

With $\boxed{\rho_0 \, dV = \rho \, J \, dV}$ we find

$$\boxed{J = \frac{\rho_0}{\rho}} \tag{2.29}$$

When mass flow is involved; the material time derivative of the mass balance must vanish,

$$\frac{D}{Dt} (J\rho - \rho_0) = 0 \tag{2.30}$$

With $\frac{D\rho}{Dt} J + \frac{DJ}{Dt} \rho = 0$ we find that

$$\frac{DJ}{Dt} = J \, div \, \mathbf{v} \tag{2.31}$$

For *"incompressible"* behavior, $\frac{D\rho}{Dt} = 0.$ and thus $div \, \mathbf{v} = 0$, or $v_{i,i} = 0$. Therefore, the incompressibility condition reduces to,

$$\boxed{tr \, \mathbf{d} = 0} \tag{2.32}$$

2.4 Balance of Angular Momentum

The angular momentum involves

$$\mathbf{h}_0 = \int_{\mathcal{B}_{\mathcal{I}}} (\mathbf{x} \times \rho \dot{\mathbf{x}}) \, dv \tag{2.33}$$

where the pole is assumed to coincide with the origin $\mathbf{x}_0 = 0$. The moment of the distributed forces is

$$\mathbf{m}_0 = \int_{\mathcal{B}} (\mathbf{x} \times \mathbf{b}) dv + \int_{\partial\mathcal{B}} (\mathbf{x} \times \mathbf{t}_n) \, da \tag{2.34}$$

The balance of angular momentum states

$$\boxed{\frac{D\mathbf{h}_0}{Dt} = \mathbf{m}_0} \tag{2.35}$$

$$\frac{D}{Dt} \int_{\mathcal{B}} (\mathbf{x} \times \rho \dot{\mathbf{x}}) dv = \int_{\mathcal{B}} (\mathbf{x} \times \mathbf{b}) dv + \int_{\partial\mathcal{B}} (\mathbf{x} \times \mathbf{t}_n) \, da \tag{2.36}$$

The divergence theorem yields for the last term above

$$\int_{\mathcal{B}} [\mathbf{x} \times (\mathbf{b} + div\,\boldsymbol{\sigma}^t - \rho\ddot{\mathbf{x}})]dv + 2\int_{\mathcal{B}} (\mathbf{1} \times \boldsymbol{\sigma}^t)dv \;=\; 0 \tag{2.37}$$

Application of the first theorem of Cauchy : $\mathbf{b} + div\,\boldsymbol{\sigma}^t - \rho\,\frac{D}{Dt}\dot{\mathbf{x}} \;=\; 0$, we get

$$\boxed{\mathbf{1} \times \boldsymbol{\sigma}^t = \mathbf{0}} \rightarrow \boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^t} \tag{2.38}$$

which states that the stress tensors symmetric, i.e. $\boldsymbol{\sigma}^t = \boldsymbol{\sigma}$, i.e. the *"Boltzmann Postulate"* of a symmetric stress tensor.
In index notation :

$$e_{ijk}\sigma_{jk} = 0 \rightarrow \sigma_{jk} = \sigma_{kj} \tag{2.39}$$

this results in

$$\sigma_{23} - \sigma_{32} = 0;\ \sigma_{31} - \sigma_{13} = 0;\ \sigma_{12} - \sigma_{21} = 0 \tag{2.40}$$

2.5 Alternative Stress Measures:

Consider that the force vector is the same on the deformed and undeformed surface areas

$$\mathbf{f} \;=\; \mathbf{t}_n\,da \;=\; \boldsymbol{\sigma}^t\,\mathbf{n}\,da = \boldsymbol{\Sigma}^t\,\mathbf{N}\,dA \tag{2.41}$$

then $\boldsymbol{\Sigma}$ denotes the *"First Piola-Kirchhoff"* stress tensor with respect to the undeformed surface area.

From Nanson's formula $\boxed{\mathbf{n}\,da \;=\; J\,\mathbf{F}^{-t}\,\mathbf{N}\,dA}$ we get

$$J\,\boldsymbol{\sigma}^t\,\mathbf{F}^{-t}\,\mathbf{N}\,dA \;=\; \boldsymbol{\Sigma}^t\,\mathbf{N}\,dA \tag{2.42}$$

or

$$\boxed{\boldsymbol{\Sigma} \;=\; J\,\mathbf{F}^{-1}\,\boldsymbol{\sigma}} \tag{2.43}$$

which shows the loss of symmetry of the first Piola-Kirchhoff stress, $\boxed{\boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}^t}$.

The *"Second Piola-Kirchoff"* stress is defined as

$$\mathbf{S} \;=\; \boldsymbol{\Sigma}\,\mathbf{F}^{-t} \;=\; J\,\mathbf{F}^{-1}\,\boldsymbol{\sigma}\,\mathbf{F}^{-t} \tag{2.44}$$

or

$$\mathbf{S} \;=\; \mathbf{F}^{-1}\,\boldsymbol{\tau}\,\mathbf{F}^{-t} \tag{2.45}$$

in which $\boxed{\boldsymbol{\tau} \;=\; J\,\boldsymbol{\sigma}}$ denotes the Kirchoff stress. From $\boxed{\boldsymbol{\sigma} \;=\; \frac{1}{J}\,\mathbf{F}\,\mathbf{S}\,\mathbf{F}^t}$, we can relate the Kirchhoff stress to the second Piola-Kirchhoff stress

$$\boldsymbol{\tau} \;=\; \mathbf{F}\,\mathbf{S}\,\mathbf{F}^t \tag{2.46}$$

Balance of Linear Momentum:

In the current reference configuration

$$\frac{D}{Dt}\left(\int_{\mathcal{B}} \rho\, \dot{\mathbf{x}}\, dv\right) \;=\; \int_{\mathcal{B}} \mathbf{b}\, dv \;+\; \int_{\partial\mathcal{B}} \mathbf{t}_n\, da \tag{2.47}$$

in which $\mathbf{t}_n \;=\; \boldsymbol{\sigma}^t \cdot \mathbf{n}$. In the initial undeformed reference configuration this reads

$$\int_{\mathcal{B}_0} \rho_0\, \frac{D\dot{\mathbf{x}}}{Dt}\, dV \;=\; \int_{\mathcal{B}_0} \mathbf{b}_0\, dV \;+\; \int_{\partial\mathcal{B}_0} \boldsymbol{\Sigma}^t\, \mathbf{N}\, dA \tag{2.48}$$

2.6 Mechanical Stress Power:

Conjugate forms of kinematic and static measures lead to an inner product form of stress and deformation rate. Using the divergence theorem the spatial description of momentum balance leads to the local statement of differential equilibrium :

$$\boxed{div\, \boldsymbol{\sigma}^t \;+\; \mathbf{b} \;=\; \rho\, \tfrac{D\mathbf{v}}{Dt}} \tag{2.49}$$

If this equation of motion is scalar multiplied with $\mathbf{v}$ and integrated over the entire body, we get

$$\int_{\mathcal{B}} (div\, \boldsymbol{\sigma}^t) \cdot \mathbf{v}\, dv \;+\; \int_{\mathcal{B}} \mathbf{b} \cdot \mathbf{v}\, dv \;=\; \int_{\mathcal{B}} \rho\, \dot{\mathbf{v}} \cdot \mathbf{v}\, dv \tag{2.50}$$

With $(div\, \boldsymbol{\sigma}^t) \cdot \mathbf{v} \;=\; div(\boldsymbol{\sigma}^t \cdot \mathbf{v}) \;-\; \boldsymbol{\sigma} : \mathbf{d}$, we get

$$\int_{\mathcal{B}} \mathbf{b} \cdot \mathbf{v}\, dv \;+\; \int_{\partial\mathcal{B}} \mathbf{t} \cdot \mathbf{v}\, da \;=\; \int_{\mathcal{B}} \boldsymbol{\sigma} : \mathbf{d}\, dv \;+\; \frac{D}{Dt} \int_{\mathcal{B}} \rho \dot{\mathbf{v}}.\mathbf{v}\, dv \tag{2.51}$$

Using the divergence theorem the material description of momentum balance leads to the local statement of differential equilibrium :

$$\boxed{Div\, \boldsymbol{\Sigma}^t \;+\; \mathbf{b}_0 \;=\; \rho_0\, \tfrac{D\,\mathbf{V}}{Dt}} \tag{2.52}$$

If this equation is multiplied with the weighing function $\mathbf{v}$ and integrated over the entire body in the reference configuration, we find

$$\int_{\mathcal{B}_0} (Div\, \boldsymbol{\Sigma}^t) \cdot \mathbf{v}\, dV \;+\; \int_{\mathcal{B}_0} \mathbf{b_0} \cdot \mathbf{v}\, dV \;=\; \int_{\mathcal{B}_0} \rho_0\, \frac{D}{Dt}\, \dot{\mathbf{V}} \cdot \mathbf{v}\, dV \tag{2.53}$$

After analogous calculation to the spatial description we get

$$\int_{\mathcal{B}_0} \mathbf{b}_0 \cdot \mathbf{v}\, dV \;+\; \int_{\partial\mathcal{B}_0} (\boldsymbol{\Sigma}^t \cdot \mathbf{N}) \cdot \mathbf{v}\, dA \;=\; \int_{\mathcal{B}_0} \boldsymbol{\Sigma}^t : \dot{\mathbf{F}}\, dV \;+\; \int_{\mathcal{B}_0} \rho_0\, \dot{\mathbf{V}} \cdot \mathbf{v}\, dV \tag{2.54}$$

Stress Power:

Considering $\boldsymbol{\tau} \;=\; \rho\, \boldsymbol{\sigma}$, the inner product of stress and the rate of deformation leads to alternative representations in terms of conjugate values

$$\dot{W}_\sigma \;=\; \frac{1}{\rho_0}\, \boldsymbol{\tau} : \mathbf{d} \;=\; \frac{1}{\rho}\, \boldsymbol{\sigma} : \mathbf{d} \;=\; \frac{1}{\rho_0}\, \boldsymbol{\Sigma}^t : \dot{\mathbf{F}} \;=\; \frac{1}{\rho_0}\, \mathbf{S} : \dot{\mathbf{E}}_G \tag{2.55}$$

Chapter 3

Lagrangian and Eulerian Descriptions

Abstract

This section develops Lagrangean and Eulerian settings for the finite element description of motion in terms of linearized incremental statements of virtual work.

3.1 Strong Equilibrium Statements

Considering the balance of momentum neglecting inertia effects:

1. Reference configuration:

$$\boxed{Div\,\mathbf{\Sigma}^t \;+\; \mathbf{b}_0 \;=\; \mathbf{0}} \tag{3.1}$$

   and the traction boundary condition $\mathbf{T} \;=\; \Sigma^t \cdot \mathbf{N}$ in the reference configuration; The balance of angular momentum results in symmetry of $\mathbf{F} \cdot \mathbf{\Sigma} \;=\; \mathbf{\Sigma}^t \cdot \mathbf{F}^t$

2. Current configuration:

$$\boxed{div\,\boldsymbol{\sigma}^t \;+\; \mathbf{b} \;=\; \mathbf{0}} \tag{3.2}$$

   and the traction boundary condition $\mathbf{t} \;=\; \boldsymbol{\sigma}^t \cdot \mathbf{n}$ in the current configuration. The balance of angular momentum results in symmetry of the Cauchy stress tensor, i.e. $\boldsymbol{\sigma} \;=\; \boldsymbol{\sigma}^t$.

3.2 Weak Equilibrium Statements: $\delta W_i \;= \delta W_e$

The weak form of equilibrium is expressed in terms of the virtual work principle, with the internal work being defined in the reference configuration as

$$\delta W_i \;=\; \int_{\mathcal{B}_0} Grad(\delta\mathbf{u}) : \mathbf{\Sigma}\, dV \tag{3.3}$$

and in the current configuration as

$$\delta W_i \;=\; \int_{\mathcal{B}} grad(\delta\mathbf{u}) : \boldsymbol{\sigma}\, dv \tag{3.4}$$

Two descriptions will be disscussed for linearizing these weak equilibrium statements:

1. The Lagrangian Approach: Total Lagrange Method.

2. The Eulerian Approach: Updated Lagrange Method.

3.3 Lagrangian Approach:

The incremental form of weak equilibrium derives directly from the weighted form of point-wise balance of incremental linear momentum. In the reference configuration this results in the increment of virtual work $\delta\dot{U} = \delta\dot{W}$ with

$$\delta\dot{W}_i = \int_{\mathcal{B}} Grad(\delta\mathbf{u}) : \dot{\mathbf{\Sigma}}\, dV \tag{3.5}$$

Recall that $\dot{\mathbf{F}} = Grad(\dot{\mathbf{u}}) = \boldsymbol{\ell}\cdot\mathbf{F}$, and that the tangential material law in the reference configuration is defined as $\dot{\mathbf{S}} = \mathbf{C}_0 : \dot{\mathbf{E}}_G$,

With $\frac{D}{Dt}(Div\,\mathbf{\Sigma}) = Div\,\dot{\mathbf{\Sigma}}$, and with $\mathbf{\Sigma} = \mathbf{S}\cdot\mathbf{F}^t$ we may substitute these expressions into the incremental form of the internal virtual work, which leads to

$$\delta\dot{W}_i = \int_{\mathcal{B}} Grad(\delta\mathbf{u}) : (\dot{\mathbf{S}}\cdot\mathbf{F}^t + \mathbf{S}\cdot\dot{\mathbf{F}}^t)dV = \int_{\beta_1} Grad(\delta\mathbf{u}) : (\mathbf{F}\cdot\mathbf{C_0} : \dot{\mathbf{E}}_G + \mathbf{S}\cdot\mathbf{F}^t\cdot\boldsymbol{\ell}^t)\, dV \tag{3.6}$$

where $\dot{\mathbf{E}}_G = \frac{1}{2}(\dot{\mathbf{F}}^t\cdot\mathbf{F} + \mathbf{F}^t\cdot\dot{\mathbf{F}})$.

This expression may be decomposed into two separate terms,

$$\begin{aligned}\delta\dot{W}_i \;=\; & \int_{\mathcal{B}} Grad(\delta\mathbf{u}) : \mathbf{S}\cdot Grad(\dot{\mathbf{u}})^t\, dV \;+ \\ & \frac{1}{2}\int_{\mathcal{B}} Grad(\delta\mathbf{u}) : \mathbf{F} : \mathbf{C}_0 : (\mathbf{F}^t\cdot Grad(\dot{\mathbf{u}}) + Grad\,(\dot{\mathbf{u}}^t)\mathbf{F})dV\end{aligned} \tag{3.7}$$

in which the term $Grad(\delta\mathbf{u}) : \mathbf{S}\cdot Grad(\dot{\mathbf{u}})^t$ gives rise to the *"geometric stiffness"*, and the second term $Grad(\delta\mathbf{u}) : \mathbf{F}\cdot\mathbf{C}_0 : (\mathbf{F}^t\cdot Grad(\dot{\mathbf{u}}) + Grad(\dot{\mathbf{u}})\,\mathbf{F})$ gives rise to the *"material stiffness"*

$$\delta\dot{W}_G = \int_{\mathcal{B}} Grad(\delta\mathbf{u}) : \mathbf{S}\cdot Grad(\dot{\mathbf{u}})^t\, dV \tag{3.8}$$

$$\delta\dot{W}_M = \frac{1}{2}\int_{\beta_1} Grad(\delta\mathbf{u}) : \mathbf{F} : \mathbf{C}_0 : (\mathbf{F}^t\cdot Grad(\dot{\mathbf{u}}) + Grad(\dot{\mathbf{u}})^t\cdot\mathbf{F})\, dV \tag{3.9}$$

3.4 Eulerian Approach:

We start from the definition of *"Nominal Stress"* in terms of the first Piola-Kirchhoff stress $\mathbf{\Sigma} = J\,\mathbf{F}^{-1}\boldsymbol{\sigma}$. The material time derivative involves three terms because of the chain rule of differentiation

$$\frac{D}{Dt}(\mathbf{\Sigma}) = \dot{\mathbf{\Sigma}} = \dot{J}\,\mathbf{F}^{-1}\cdot\boldsymbol{\sigma} + J\,\dot{\mathbf{F}}^{-1}\cdot\boldsymbol{\sigma} + J\,\mathbf{F}^{-1}\cdot\dot{\boldsymbol{\sigma}} \tag{3.10}$$

with the definition $\dot{J} = J\,(tr\boldsymbol{\ell})$ and with $\mathbf{F} \cdot \mathbf{F}^{-1} = \mathbf{1}$, we obtain $\dot{\mathbf{F}}^{-1} = -\mathbf{F}^{-1} \cdot \boldsymbol{\ell}$ .The material derivative of the nominal stress is henceforth

$$\dot{\boldsymbol{\Sigma}} = J\,\mathbf{F}^{-1}\,((tr\boldsymbol{\ell})\,\boldsymbol{\sigma} - \boldsymbol{\ell}\cdot\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}) \tag{3.11}$$

The nominal stress rate in the *"Current Configuration"* is obtained by push-forward operation and updating the reference configuration to the current configuration.

$$\dot{\boldsymbol{\Sigma}}^c = (tr\boldsymbol{\ell})\,\boldsymbol{\sigma} - \boldsymbol{\ell}\,\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} \tag{3.12}$$

since $\mathbf{F} = \mathbf{1}$ and $J = 1$.

Adopting an *"objective"* description of the hypo-elastic material law:

$$\boxed{\breve{\boldsymbol{\sigma}} = \mathbf{c} : \mathbf{d}} \tag{3.13}$$

in terms of the objective rate of deformation and an objective stress rate such as the co-rotational Jaumann rate of Cauchy-stress

$$\breve{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{w}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{w}^t \tag{3.14}$$

with the spin defined as $\mathbf{w}^t = -\mathbf{w} = -\frac{1}{2}(\boldsymbol{\ell} - \boldsymbol{\ell}^t)$, we obtain for the nominal stress rate an expansion as follows

$$\boxed{\dot{\boldsymbol{\Sigma}}^c = \breve{\boldsymbol{\sigma}} + (tr\boldsymbol{\ell})\boldsymbol{\sigma} - \boldsymbol{\ell}\cdot\boldsymbol{\sigma} + \mathbf{w}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\mathbf{w}^t} \tag{3.15}$$

or

$$\dot{\boldsymbol{\Sigma}}^c = \mathbf{c} : \mathbf{d} + (tr\boldsymbol{\ell})\,\boldsymbol{\sigma} - \boldsymbol{\ell}\cdot\boldsymbol{\sigma} + \frac{1}{2}(\boldsymbol{\ell}\cdot\boldsymbol{\sigma} - \boldsymbol{\ell}^t\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\boldsymbol{\ell}^t - \boldsymbol{\sigma}\cdot\boldsymbol{\ell}) \tag{3.16}$$

With $\boldsymbol{\ell} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}$ and $\mathbf{F}^{-1} = \mathbf{1}$, we may develop the desired relationship between the nominal stress state $\dot{\boldsymbol{\Sigma}}^c$ and the rate of the deformation gradient $\dot{\mathbf{F}} = \boldsymbol{\ell}\cdot\mathbf{F}$.

$$\boxed{\dot{\boldsymbol{\Sigma}}^c = \mathbf{D}^c_T : \boldsymbol{\ell}} \tag{3.17}$$

The *"effective"* tangential material tensor $\mathbf{D}^c_T$ is comprised of two terms which involve the *"tangential moduli"* tensor $\mathbf{c}$ and the current stress state $\boldsymbol{\sigma}$.

Explicit format :

$$D^c_{ijkl} = c^T_{ijkl} + \sigma_{ij}\delta_{kl} - \delta_{ik}\sigma_{lj} + \frac{1}{2}(\delta_{ik}\sigma_{lj} - \delta_{il}\sigma_{kj} + \delta_{jk}\sigma_{il} - \delta_{jl}\sigma_{ik}) \tag{3.18}$$

or expanded

$$\dot{\Sigma}^c_{ij} = c^T_{ijkl}\,d_{kl} + \sigma_{ij}\,\ell_{kk} - \ell_{il}\sigma_{lj} + \frac{1}{2}(\ell_{il}\sigma_{lj} - \ell_{ik}\sigma_{kj} + \ell_{jl}\sigma_{il} - \ell_{jk}\sigma_{ik}) \tag{3.19}$$

Substituting the rate of the nominal stress tensor into the weak equilibrium statement, the virtual internal work statement in the current configuration reduces to

$$\delta \dot{W}_i \;=\; \int_{\beta_1} Grad(\delta \mathbf{u}) : \dot{\mathbf{\Sigma}}\, dV \rightarrow \int_{\beta} grad(\delta \mathbf{u}) : \dot{\mathbf{\Sigma}}^c\, dv \tag{3.20}$$

with $\boldsymbol{\ell} = grad(\dot{\mathbf{u}})$, the incremental virtual work expands into

$$\delta \dot{W} \;=\; \int_{\beta} grad(\delta \mathbf{u}) : \mathbf{D}_T^c : \boldsymbol{\ell}\, dv \;=\; \int_{\beta} grad(\delta \mathbf{u}) : \mathbf{D}_T^c : grad(\dot{\mathbf{u}})\, dv \tag{3.21}$$

This incremental equilibrium statement gives rise to the *" Tangential Material Stiffness"* and the *"Geometric Stiffness"* operators defined by:

$$\boxed{\delta \dot{W}_i \;=\; \delta \dot{W}_M \;+\; \delta \dot{W}_G} \tag{3.22}$$

With the tangential material law $\breve{\boldsymbol{\sigma}} \;=\; \mathbf{c} : \mathbf{d}$, we recover the material stiffness operator of the small displacement theory

$$\delta \dot{W}_M \;=\; \int_{\beta} grad(\delta \mathbf{u}) : \mathbf{c} : \mathbf{d}\, dv \;=\; \int_{\beta} \delta \boldsymbol{\epsilon} : \mathbf{c} : \mathbf{d}\, dv \tag{3.23}$$

and the geometric stiffness of the initial stress operator

$$\delta \dot{W}_G \;=\; \int_{\beta} grad(\delta \mathbf{u}) : \hat{\boldsymbol{\sigma}} : grad(\dot{\mathbf{u}})\, dv \tag{3.24}$$

In the finite element sense, these virtual work terms translate into matrix notation:

$$\delta \dot{W}_i^h \;=\; \delta \mathbf{u}_I^t\, (\mathbf{k}_E \;+\; \mathbf{k}_G)\, \dot{\mathbf{u}}_I \tag{3.25}$$

and

$$\delta \dot{\mathbf{W}}_e^h \;=\; \delta \mathbf{u}_I^t\, \dot{\mathbf{f}}_I \tag{3.26}$$

The *"Tangential Stiffness"* relationship describes a quasi-linear statement of incremental equilibrium which reads at the element level as;

$$\boxed{(\mathbf{k}_E \;+\; \mathbf{k}_G)\, \dot{\mathbf{u}}_I \;=\; \dot{\mathbf{f}}_I} \tag{3.27}$$

With $\mathbf{G} \;=\; \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$, and the material tangent $\mathbf{E}_T \;=\; \mathbf{c}$, we retrieve the format of the Eulerian tangent stiffness operators,

$$\mathbf{k}_E \;=\; \int_{\Omega_e} \mathbf{B}_L^t\, \mathbf{E}_T\, \mathbf{B}_L\, dv \tag{3.28}$$

$$\mathbf{k}_G \;=\; \int_{\Omega_e} \mathbf{G}^t\, \hat{\boldsymbol{\sigma}}\, \mathbf{G}\, dv \tag{3.29}$$

in which $\mathbf{B}_L$ denotes the traditional linear rate of deformation-nodal velocity operator and $\mathbf{G}$ the spatial gradient of the finite element velocity expansion.